# Quantitative Methods in Economics
## (Multiple Criteria Decision Making XV)

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# QUANTITATIVE METHODS IN ECONOMICS
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# DISPATCHING OF MOVABLE HANDLING DEVICES

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**Abstract:** In this paper operational control strategies for movable material handling devices are analyzed. To analyse effects of implementing five dispatching rules simulation model in Flexsim 3 has been developed.

**Keywords:** Operational control, materials handling systems, dispatching rules, simulation.

## 1. INTRODUCTION

Because of huge impact that material handling operations have on transportation and warehousing systems its rationalization is of crucial importance in providing efficient logistics operations. Therefore, appropriate strategies to control material handling devices are always necessary, but its implementation requires comprehensive analysis of effects they make on observed system. In that sense, the objective of this paper is to analyze effects and possibilities of implementation of different operational level control strategies on a material handling system based on movable devices, which serve tasks distributed in space and time. As an example of movable handling devices we considered system for unloading gravel from river barges.

## 2. PROBLEM DESCRIPTION AND LITERATURE REVIEW

The problem analyzed in this paper considers operational control of devices for unloading gravel from barges. Namely, gravel distribution process implies gravel drawing from riverbed, its loading into barges and transport to some of several unloading locations. Devices used for unloading process comprises vessel, crane and belt conveyor. Therefore, this technology is able to provide unloading service at any place on a river bank, i.e. wherever unloading task occurs. However, because the number of these devices, due to their high costs, is lower than the number of potential unloading locations in those systems arises operational control problem of determining sequence in which tasks should be served within considered planning horizon. Because barges' owners are usually at the same time owners of unloading devices, the objective of the problem is to minimize tasks waiting time, which is equivalent to the minimization of time that barges wait until they are served.

This type of problem can be considered either as a static or a dynamic. Advantage of considering it as a static problem is that optimal control decisions may be done by implementing exact solution methods. Unfortunately this is possible only for relatively small problem instances. Solving the static problem was subject of paper [1] where authors introduced two mathematical formulations of this problem which is called Handling Devices Allocation Problem – HDAP. First formulation considers HDAP as a three dimensional assignment problem, while the second is based on similarities with Static Berth Allocation Problem – SBAP but with respecting differences between them. Beside that, this paper presents three-step heuristics call CLASORD (CLustering ASsignment ORDering) for special case of large static HAD problems.

Practical implementation of static approaches is possible only if all relevant data about the system are known prior to the start of decision making process and if there are no changes in that data during the process realization, i.e. when system is not stochastic in its nature. However, since systems of this type are usually stochastic due to influence of different factors (weather conditions, transportation and handling devices' breakdowns, occurrence of priority customers etc.) it is obvious that practical implementation of static approach is limited.

On the other side, dynamic approach to control implies that decisions are made during the process realization meaning that decisions are based on up-to-date information about the system state and at the same time making practical implementation easier. The most often implementation of dynamic control is based on dispatching, i.e. on dispatching rules defined in advance.

Dispatching implies that rules are triggered when one of two events happen in the system. The first event is occurrence of new task in the system, which is called task driven dispatching. The second event is end of serving a task, i.e. moment when device become available for service of another task. This type of dispatching is called device driven dispatching. In the first case it is required to choose one device from set of available devices, while the second requires choosing one from a set of tasks waiting in the system to be served. In both cases selection is realized accordingly to criteria in predefined set of rules. Since operation of a real system implies occurrences of both types of dispatching it is obvious that rules of both types must be implemented as a tool for dynamic control of a system. However, the largest drawback of dispatching is lack of planning horizon when decisions about device allocation are made. In other words, decision is made without getting insight into consequences.

Use of dispatching rules in material handling problems is related to internal transportation system, primarily to systems based on AGV. One of fundamental papers regarding this type of problems is [2] in which detailed description of basic rules is presented as well as distinction of rules on either task or device driven ones. Development of AGV systems was followed up with development of implemented dispatching rules. Firstly by introducing additional criteria in decision rules, i.e. by implementing multicriteria rules [3], after that by implementing contemporary decision support system based on fazzy logic [4] and neural networks, and finally huge attention was paid to a class of so called “intelligent” rules which are based on possibility of reassigning already assigned tasks if that is going to improve objective function. Use of “intelligent” rules [5] in some degree overcome mentioned drawback of traditional rules.

In any case, prior to deciding which rule should be implemented it is necessary to determine resulting performances of applying certain rule on an exact system.

## 3. CONTROL OF GRAVEL UNLOADING DEVICES BY DISPATCHING RULES

In this paper we analyzed performance of five dispatching rules used for control of system for unloading gravel from barges. Implemented rules are:

- First Come First Served – FCFS
- Fastest Realization First – FRF
- Multi Criteria – MC
- Bidding Based - BB
- Bid Based Dynamic Dispatching – $B^2D^2$

First two rules belong to a class of single criteria rules, third and fourth rule to a class of multi criteria rules, while fifth rule belongs to a class of “intelligent” rules.

FCFS rule is device driven rule which implies that, of all unassigned tasks, tasks that first came in the system is selected, regardless to distance from device that initiated rule execution.

FRF rule is universal one, meaning that it is both: task and device driven. Its implementation implies that task whose service is going to be finished first, or device which is going to serve task fastest are selected, depending on a cause of rule initialization (task or device). This rule in its essence is modification of widely implemented and efficient rules Nearest Vehicle First – NVF and Shortest Travel Distance (Time) First – STD(F)F.

MC is also universal rule whose assignment decisions, in opposite to previous ones, are based on more than one aspect of the problem, i.e. on more than one criteria. Generally, decisions are made by implementing any method for solving multicriteria decision making problems. For the case of this paper weight function method is used and criteria that are considered are: distance between locations of device and task involved, tasks' waiting times and overall quantities of gravel waiting for unloading within locations.

BB rule is, like MC, universal rule which take into consideration more than one criteria but in this case by implementing bid based concept. Namely, with each occurrence of triggering event each device, or task, place its bid representing its suitability for allocation. Bid values are calculated according to predefined equations. In this case following equations are used to calculate bid values:

$$f_a(a) = (1-b)\cdot a + b \quad 0 \le a \le 1, 0 \le b \le 1 \qquad (1)$$

$$f_t(t) = t^{\alpha} \qquad (2)$$

$$f_q(q) = q^{\beta} \qquad (3)$$

where a, t and q are distance between locations of device and task, time task waits in the system and overall quantity of gravel for unloading within locations, respectively. Coefficients, b, $\alpha$ and $\beta$ affecting the shape of bid functions. Final assignment decision is made according to values of dispatching functions that merges influences of all bid values. Since all mathematical operands are allowed in defining dispatching functions number of defferent functions is very large. In this paper following dispatching functions are used:

$$D_1 = f_t + f_q - f_a \tag{4}$$

$$D_2 = f_t \cdot f_q / f_a \tag{5}$$

$$D_3 = f_t \cdot f_q \cdot (1 - f_a) \tag{6}$$

$B^2D^2$ is task driven rule which differs from previous in allowing already assigned task to be reassigned to other device in case that it is going to reduce task's serving time. Algorithm of this rule is presented in figure 1. It should be noted that its power actually lies in the operating mechanism that provides postponement of task's final assignment by introducing temporary assignment until more data about system status are available. Simplified procedure can be described in following way. When task occurs in a system each device place a bid representing time when device will finish task's service after serving all previously finally assigned tasks. If this bid is lower then the thread value, task is finally assigned to device with lowest bid value. Otherwise, it is temporary assigned to a device with lowest bid value with possibility to be reassigned to another device with new task occurrence if that is going to reduce its task's service ending times.

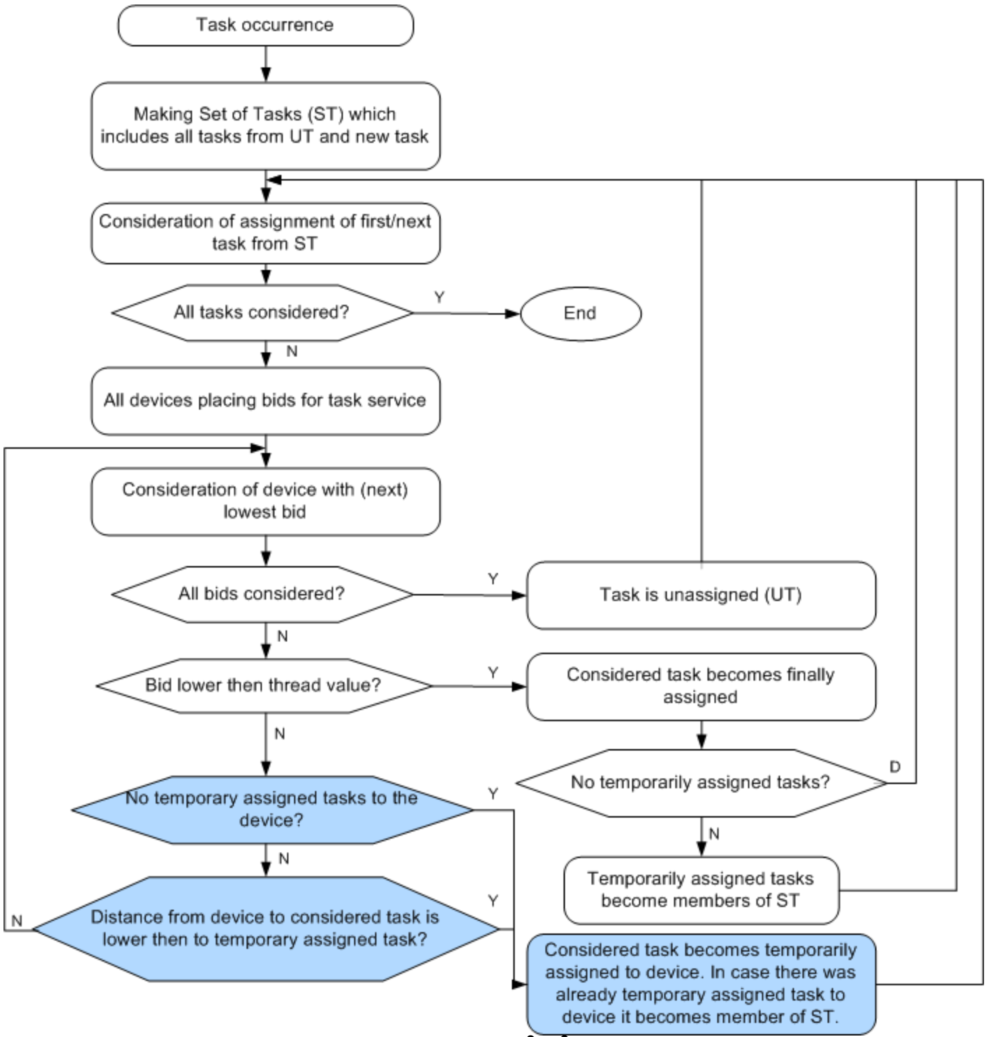


Figure 1: Algorithm of $B^2D^2$ dispatching rule

**4. NUMERIC EXAMPLE**

Considered system comprises three handling devices and seven locations whose spatial distribution is presented on figure 2. Devices' speeds are 5 km/h and capacities are 100, 150 and 200t/h, respectively. At the beginning of planning horizon all devices are located at depot.

Because operation of this system implies implementation of both task and device driven dispatching rules and since some of them are only device driven, our experiment considered organization in which FRF was implemented for all experiments as task driven rule. This is not the case only for the $B^2D^2$ rule which is task driven only. This organization does not have significant influence on overall performances because, due to big task arrival intensity, task driven rules are triggered only at the beginning of planning horizon. In that sense, during analyzed planning horizon of 48h tasks arrival rate is taken to be 40% more intensive then capacity of available resources. Tasks are generated according to uniform distribution on (3, 5.4) hour interval, i.e. case of 12 tasks during planning horizon is considered. Task locations are also uniformly distributed on all seven locations and each task implies unloading of 1000t of gravel.

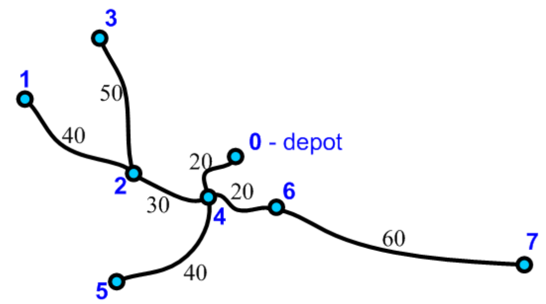


Figure 2: Spatial distribution of gravel unloading locations

## 5. SIMULATION RESULTS

To analyse defined dispatching rules on presented system, simulation model implemented in Flexsim 3 simulation package has been developed. Simulation results in form of average and standard deviation of overall tasks' waiting time are shown in table1.

Table1: Simulation results

| | FCFS | FRF | MC | BB | $B^2D^2$ | $B^2D^2$ without TH |
|---|---|---|---|---|---|---|
| Average [h] | 365.24 | 291.04 | 289.44 | 288.22 | 253.55 | 281.96 |
| St. deviation [h] | 44.04 | 14.13 | 16.40 | 45.48 | 35.52 | 51.89 |

From results it can easily be seen domination of $B^2D^2$ rule compared to other rules. Also, in case of multicriteria rules BB rule outperformed MC. Beside that, although MC considers more than one criteria unlike FRF, later rule gave better results in this case. That result is in accordance with previously conducted researches in which FRF rule proved its goodness, even when compared with more advanced rules. Good performances, beside its simplicity, is the main reason for wide spread use of FRF rule.
Table 2 contains column denoted with $B^2D^2$ without TH which shows result of implementation of a modified $B^2D^2$ rule. Namely, this modification analyses effects of implementation of $B^2D^2$ rule when there is no thread value, i.e. when algorithm presented on figure 1 is applied without grayed blocks. From results it can be seen that even this “diminished intelligent” rule outperformed all other “non intelligent” rules, except BB rule.

## 6. CONCLUSION

So far HDAP is analyzed for cases of static control and dynamic control with zero planning horizon. Control based on dynamic control with non-zero planning horizon has not been analyzed yet ad therefore that is direction in which we plan to continue our research related to this problem.

## ACKNOWLEDGEMENT

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# REGIONAL MODELING AT THE SLOVAK ACADEMY OF SCIENCES

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**Abstract:** Regional cohesion policy has been presented as one of the main priorities of the European Union and also most of its member-states including Slovakia as well. Regional disparities seem complicated not only from the national perspective, but often more as international problems. This is also a problem of the Slovak republic, where regional disparities at the NUTS3 level are greater than at the NUTS2 level. Tackling regional disparities and promoting cohesion are the main objectives of the Slovak cohesion policy and are also included in The Manifesto of the Government of the Slovak Republic. In this paper, a modeling approach is presented. The regional model of the Institute of Economic Research at the Slovak Academy of Sciences B_IER_REG, which is connected to the econometric model B_IER_ECM allows to analyze and forecast different approaches to Slovak regional cohesion policy and their influence on the regions of Slovakia. The aim of this paper is to introduce the model B_IER_REG and to describe its main characteristics.

**Keywords:** Regional modeling, Regional disparities, Regional analysis.

## 1. Introduction

The regional cohesion policy is presented in the Slovak republic as one of the long-term priorities of national government and is included in the government program. One of the main objectives of the aforementioned cohesion policy is balancing regional disparities. We use the regional model of the Institute of Economic Research at the Slovak Academy of Sciences in Bratislava B_IER_REG, which is connected to the econometric model B_IER_ECM and allows us to analyze and forecast different approaches to Slovak regional cohesion policy and their influence on the regions of Slovakia.

## 2. Regional model of the Slovak Republic

Regional development in Slovakia is influenced by a set of social and economic considerations. The regional model of the Slovak republic is founded on a so called top-down principle with a feedback to the block of GDP and based on our macro-economic forecasts from the econometric model B_IER_ECM_09q3. The aggregate macroeconomic indicators from the econometric model B_IER_ECM_09q3 are inputs for the regional model as certain restrictions, respectively. This model is based on quarterly data from the first quarter of 1995 to the second quarter of 2009, (58 observations). The medium-term forecast provides us a view until the year 2020. The sources of data are:

- The Statistical Office of Slovakia,
- The National Bank of Slovakia,
- The Ministry of Labor,
- The Ministry of Social Affairs and Family

- The Ministry of Finance.

The economic development analysis of the regions confirmed the facts already known to the scientific community and provide us some further information about the changes and the trends in their development throughout the past 10 years. The regional model is divided into the following sections:

- Demographics,
- Labor market
- Investment,
- Exogenous and expert inputs (qualitative analysis of the expected regions and regional policy).

The regional model of the Slovak Republic B_IER_REG_09 is based on basic economic relationships (see Figure 1) and takes into account both the supply side and the demand side. This model is based on annual data for the last 10 years. The main characteristics of their blocks are as follows.

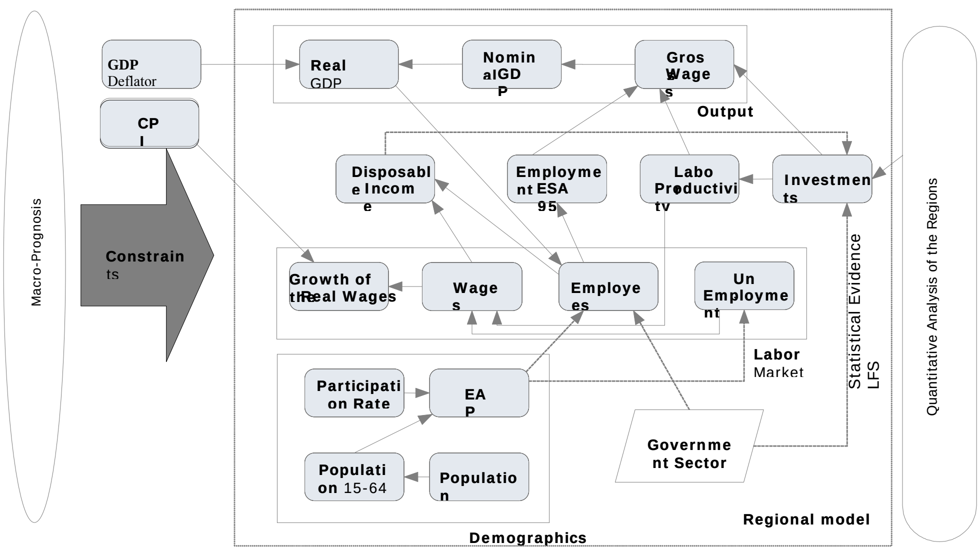
The demographic block, which designates the human capital stock and labor supply on the basis of middle-term forecast. Both indicators are usually the same for all variants. The demographic forecast constitutes an important part in our regional model because its results strongly determine the results of the forecast from the regional model. The main output of this block is the forecast of the economically active population, which significantly determines the development of the labor market. Forecast of the population is based on the information from the Demographic Research Centre (Vaňo, 2008).

The block of the labor market simulates employment, unemployment and wages. They are modeled statistical indicators both on the basis of statistical evidence (registrations) and on the basis of the Labor Force Survey (LFS).

The gross output is modeled using employment ESA95 and labor productivity. Furthermore, the productivity is influenced by the level of investments. The level of nominal GDP is calculated on the basis of gross production and the real GDP is calculated from nominal GDP using GDP deflator.

Exogenous and expert inputs are the qualitative indicators of the regions (current state analysis, analysis of the potential of the regions and future development analysis) and the expected regional policy (the sector of general government). The main results of the regional forecast are real GDP, real wages and basic labor market indicators.

**Figure 1** – The structure of the Regional model of the Slovak Republic B_IER_REG_09




Source: Authors

The changes in the model are modeled through the government sector. Usually we assume in the modeled scenarios which are created on the basis of the so called bottom-up principles changes in the structure, volume and efficiency of the public expenditures from domestic and external sources (mainly from the European union). The parameters of the model were calibrated in the baseline scenario.

## 3. Macroeconomic model of the Slovak Republic

Currently, the forecasting of future development in the world's major economies on the grounds of the global economic recession is a relatively difficult task. This makes it difficult to predict the future development of a small and highly open economy, such Slovakia is. Currently, we can say that the short-term forecast is reviewed with a monthly frequency, generally downward. The stabilization of the forecasts can be observed only for few past months. The analysts are cautious when reducing growth especially to a negative value and are prone to accept positive expectations.

The instability of the external and internal factors can determine the future development in the medium term and can induce higher risks when the prognoses are created. The baseline scenario is based on the current state of the internal and external environment and represents the most plausible economic development of Slovakia. The forecast in relation to the requirements of the EU covers the horizon of ten years from 2010 to 2020.

Different scenarios allow us the comparison of the developments in regions at the level NUTS 3 on the basis to different orientations of the regional economic policy. Therefore, the prognostic accuracy of the baseline scenario is not crucial. More important are the differences in the economic development between the aforementioned scenarios.

**Figure 2** - Slovak NUTS 3 regions


Econometric model B_IER_ECM_09q3 was designed in order to create forecast of the future development of Slovak economy. It is based on quarterly data, from 1995q1 to 2009q3, which means 59 observations available. Forecast is medium-term until the year 2015. Our sources of data are:

- The Statistical Office of the Slovak Republic,
- The National Bank of Slovakia,
- The Ministry of Finance of the Slovak Republic.

The model is demand-oriented and bases on main macroeconomic principles. The model consists of 5 main blocks which are as follows:

- Labor market block,
- State budget block,
- Block of prices,
- Foreign trade block.

- GDP block.

Model contains 52 equations. While 31 of these are stochastic and the remaining 21 are identities. Stochastic equations are based on the ECM principles (error-correction method). In particular equations, integrated time series of the same order are used and long-term equilibrium is described by the co-integration relationships. The structure of this model is described in (Radvansky, 2008).

## 4. Conclusion

As regional disparities at the NUTS3 level represent a crucial issue, regional cohesion policy represents one of the main priorities of the Slovak government.The regions in the western part of Slovakia are economically stronger (Bratislava region, Trnava region) than the regions in the middle or eastern part of the country (Banská bystrica region, Prešov region, Košice region). The main objective of the government is to balance these disparities. In this paper, we presented the regional model of Slovakia at the NUTS 3 level ( so-called counties) which allows us to analyze different cohesion policies and their impact on the level of the disparities and on the development of the national economy.

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# USING TOPSIS METHOD WITH LAPLACE CRITERION TO SELECT OPTIMUM AIRLINE

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**Abstract** — In this study, for evaluating subjective features that provides preference of airline companies to others the method TOPSIS has been used. Whilst calculating the weights of the criteria Laplace Criterion had been used. The importance of the study is that this is a unique application in air cargo industry.

**Keywords** — Air Cargo Shipping, Multi-Criteria Decision Making, TOPSIS, Laplace Criterion.

## 1. Introduction

According to Airport Council International data in 2007 the amount shipped by air cargo in world has been increased to 48.3 million tones by 3.6%. In Turkey the amount shipped is approximately 310.000 tones according to State Airports Administration data. By air, Turkey imported $18 billion and exported $8 billion of merchandise in 2007. When compared to last year, the value of merchandise imported had been increased by 17% while the value of merchandise exported had increased by 44% (Uticat sector report, 2007).

In this study, we extend TOPSIS to solve a multiple criteria decision making problem. The problem is selection of the best airline company between five competitive airlines. The remainder of this paper is organized as follows. Next section presents the literature available in logistics sector using TOPSIS methodology. Section 3 briefly discusses the methodology of TOPSIS. In section 4 we applied the TOPSIS method to the selection of the airline company. Conclusions are drawn in section 5.

## 2. Literature Review

When a decision maker must choose one among a number of possible actions, the ultimate consequences of some if not all of these actions will generally depend on uncertain events and future actions extending indefinitely far into the future.

Upon systematically describing the problem and recording all necessary data, judgments, and preferences, the decision maker must synthesize the information set before him/her using the most appropriate decision rules. A tool commonly used to display information needed for the decision process is a payoff matrix or decision table.

These actions represent the controllable variables in the system. The uncertain events or states of nature are represented (Raiffa, 1970), (Schlaifer, 1978).

The Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), firstly introduced by Hwang and Yoon, is a multi-criteria decision making (MCDM) methodology based on the assumption that the best alternative should be as close as possible to the ideal solution and the farthest from the negative-ideal solution (Hwang and Yoon, 1981), (Babu *et al.*, 2006).

In the last years, TOPSIS has been widely applied in literature. Since there are so many studies been made we limited our research to logistics sector. Qian and Huang had applied TOPSIS to evaluate the performance of third party logistics enterprises (3PLs) and to make decision of outsourcing the logistics services (Qian and Huang, 2008). Feng and Wang constructed a performance evaluation process for airlines with financial ratios taken into consideration. This paper uses TOPSIS method for the outranking of airlines (Feng and Wang, 2000).

The Laplace insufficient reason criterion postulates that if no information is available about the probabilities of the various outcomes, it is reasonable to assume that they are equally likely. Therefore, if there are *n* outcomes, the probability of each is 1/*n*. This approach also suggests that the decision maker calculate the expected payoff for each alternative and select the alternative with the largest value. The use of expected values distinguishes this approach from the criteria that use only extreme payoffs. This characteristic makes the approach similar to decision making under risk (Kmietowicz, and Pearman, 1981).

## 3. Topsis Methodology

The principle of TOPSIS for MCDM is that the chosen solution should have the shortest distance from the positive ideal solution as well as the longest distance from the negative ideal solution (Yoon and Hwang, 1995), (Lai *et al.*, 1994).

TOPSIS defines an index called similarity to the positive- ideal solution by combining the proximity to the positive- ideal solution and remoteness from the negative- ideal solution. Then the method chooses an alternative with the maximum similarity to the positive- ideal solution. The method is presented as a series of successive steps:

*Step 1:* ***Calculate Normalized Ratings:*** The vector normalization is used for computing $r_{ij}$, which is given as

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \quad i = 1,...,\ m \text{ ve } j = 1,...,\ n \qquad [1]$$

*Step 2:* ***Calculate Weighted Normalized Ratings:*** The weighted normalized value is calculated as

$$v_{ij} = w_j * r_{ij} \quad i = 1,...,\ m;\ j = 1,...,\ n \qquad [2]$$

where $w_j$ is the weight of the *j*. attribute.

*Step 3:* ***Identify Positive-Ideal and Negative-Ideal Solutions:*** The $A^*$ and $A^-$ are defined in terms of the weighted normalized values:

$$A^* = \{v_1^*, v_2^*, ..., v_j^*, ..., v_n^*\} = \{(\max v_{ij} \mid j \in J_1), (\min v_{ij} \mid j \in J_2) \mid \text{i} = 1, ..., \text{m}\}, \quad [3]$$

$$A^- = \{v_1^-, v_2^-, ..., v_j^-, ..., v_n^-\} = \{(\min v_{ij} \mid j \in J_1), (\max v_{ij} \mid j \in J_2) \mid \text{i} = 1, ..., \text{m}\} \quad [4]$$

where $J_1$ is a set of benefit attributes and $J_2$ is a set of cost attributes.

*Step 4:* ***Calculate Separation Measures:*** The separation between alternatives can be measured by the n-dimensional Euclidean distance. The separation of each alternative from the positive-ideal solution $A^*$, is then given by

$$s_i^* = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^*)^2}, \ i = 1, ..., \ m. \qquad [5]$$

Similarly, the separation from the negative-ideal solution, $A^-$, is given by

$$S_i^- = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^-\right)^2}, \quad i = 1,..., m. \qquad [6]$$

*Step 5:* ***Calculate Similarities to Positive-Ideal Solution:***

$$C_i^* = \frac{S_i^-}{\left(S_i^* + S_i^-\right)} \quad \mathrm{i = 1,..., m.} \quad 0 \le C_i^* \le 1. \qquad [7]$$

*Step 6:* ***Rank Preference Order:*** Choose an alternative with the maximum $C_i^*$ or rank alternatives in descending order.

## 4. Numerical Example

First of all, a questionnaire survey with freight forwarder firms had been done. These freight forwarder firms are the firms that have activities in Turkey. The evaluation process is valid for the five competitive firms which does Far & Middle East cargo shipping intensively. Far & middle east cargo shipping includes either imported or exported cargos from /to Turkey. A freight forwarder is an international trade specialist who can provide a variety of functions to facilitate the movement of cross-border shipments. These functions include, but are not limited to, booking vessel, air space, preparing relevant documentation, paying freight charges, and arranging inland transport services (Murphy and Daley, 2001). Secondly, weights of the attributes determined by the Laplace Criterion. Finally, the best airline company had been chosen using TOPSIS.

TOPSIS methodology is chosen since it is unique in the way it approaches the problem and is intuitively appealing and easy to understand. Also, in literature there is no study made about airline selection problems. Hence it is important for being the first study about this topic.

The data of five airline companies related to each attribute is given in Table 1. Also, weights of each attribute are given. Since each attribute is measured on a different scale, normalization is required. By using [1] the normalized ratings are calculated and given in Table 2. Table 3 presents the normalized rating matrix.

By using [3] and [4], positive-ideal and negative-ideal solutions are calculated. Since all chosen attributes are of benefit the positive-ideal solution consists of the largest value of each column, which is denoted by the symbol "*" in Table 3. The collection of the smallest values of each column in Table 3, which are denoted by "–", makes the negative-ideal solution.

Separation measures from $A^*$ and $A^-$ is computed by using [5] and [6], respectively. Separation measures of all are given in Table 4. Equation [7] used for to calculate all similarities to the positive- ideal solutions. The results are as in Table 5. Based on the descending order of $C_i^*$, the preference order is given as THY, Emirates, Singapore, Malaysia, and Royal Jordanian.

TABLE 1
The Data of Five Airline Companies

| | Reliability | Number of Locations | Frequency | Service Quality | Customer Satisfaction |
|---|---|---|---|---|---|
| Emirates | 0,63 | 33 | 0,50 | 0,72 | 0,63 |
| THY | 0,81 | 33 | 0,50 | 0,58 | 0,73 |
| Singapore | 0,39 | 25 | 0,33 | 0,51 | 0,46 |
| Royal Jordanian | 0,22 | 31 | 0,20 | 0,23 | 0,22 |
| Malaysia | 0,26 | 46 | 0,20 | 0,24 | 0,24 |
| **Weights** | **0.20** | **0.20** | **0.20** | **0.20** | **0.20** |

TABLE 2
The Normalized Ratings

| | Reliability | Number of Locations | Frequency | Service Quality | Customer Satisfaction |
|---|---|---|---|---|---|
| Emirates | 0,5481 | 0,4304 | 0,6024 | 0,6504 | 0,5641 |
| THY | 0,7047 | 0,4304 | 0,6024 | 0,5239 | 0,6536 |
| Singapore | 0,3393 | 0,3260 | 0,3976 | 0,4607 | 0,4119 |
| Royal Jordanian | 0,1914 | 0,4043 | 0,2410 | 0,2078 | 0,1970 |
| Malaysia | 0,2262 | 0,5999 | 0,2410 | 0,2168 | 0,2149 |

TABLE 3
The Normalized Rating Matrix

| | Reliability | Number of Locations | Frequency | Service Quality | Customer Satisfaction |
|---|---|---|---|---|---|
| Emirates | 0,1096 | 0,0861 | 0,1205 | 0,1301 | 0,1128 |
| THY | 0,1409 | 0,0861 | 0,1205 | 0,1048 | 0,1307 |
| Singapore | 0,0679 | 0,0652 | 0,0795 | 0,0921 | 0,0824 |
| Royal Jordanian | 0,0383 | 0,0809 | 0,0482 | 0,0416 | 0,0394 |
| Malaysia | 0,0452 | 0,1200 | 0,0482 | 0,0434 | 0,0430 |

TABLE 4

Separation Measures

| | $S^*$ | $S^-$ |
|---|---|---|
| Emirates | **0,0495** | **0,1549** |
| THY | **0,0423** | **0,1689** |
| Singapore | **0,1175** | **0,0791** |
| Royal Jordanian | **0,1830** | **0,0156** |
| Malaysia | **0,1721** | **0,0554** |

TABLE 5

Similarities to Positive- Ideal Solution

| | | |
|---|---|---|
| Emirates | $C^*_{Emirates}$ | 0,758 |
| THY | $C^*_{THY}$ | 0,800 |
| Singapore | $C^*_{Singapore}$ | 0,403 |
| Royal Jordanian | $C^*_{RJ}$ | 0,079 |
| Malaysia | $C^*_{Malaysia}$ | 0,243 |

## 5. Conclusion

In this study we aimed to make selection between five competitive airline companies using TOPSIS method with Laplace Criterion. The airline companies mentioned in this study do mostly Far & Middle East cargo shipping. In numerical example TOPSIS method with Laplace Criterion had been applied for solving airline company selection problem. The result shows that TOPSIS method seems to be promising. TOPSIS method can be used to airline company selection decision-making of shipping company.

A questionnaire survey with freight forwarder firms had been done at the beginning of the study to choose the attributes which will be used in TOPSIS method and the weights of each attribute are accepted to be equal regarding to the Laplace Criterion. As a result, based on the descending order of $C_i^*$, the preference order of the companies is found as THY, Emirates, Singapore, Malaysia, and Royal Jordanian.

This research can also be extended by incorporating additional selection criteria such as risk factors and environmental concerns. Different alternative methodologies such as fuzzy analytic network process, fuzzy TOPSIS and fuzzy ELECTRE can also be implemented to extend the research.

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# TERMS OF OPTIMAL CONSUMER BEHAVIOR IN THE PRODUCT MARKET OF UTILITIES INDUSTRY

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**Keywords**

Utility theory, social welfare, utilities industry, marginal utility, Kuhn-Tucker conditions

## Introduction

The isolation of a market is characteristic feature of the equilibrium models of the utilities industry. Usually products of utilities industries are for the consumer not substitutable. Then, consumer feel about utility, can be quantified in a specific way. In principle it is described as utility function, when the utilities industry product is seen as having separate and precisely formulated utility function and other goods are viewed, as consumption of calculated hypothetical goods with standardized unit price.

In this paper we analyze the behavior model of consumers in utilities industry markets and with the questions of effectiveness of this specific market. We will present the optimal conditions of Kuhn - Tucker for general non-linear programming optimization problem witch goal is to maximize the utility function of consumers in utilities industry markets with specifically constructed condition of consumer costs on production of utilities industry.

## 1. Optimization of costumer consumption strategy

The general problem of optimization of consumer's behavior as a classical category of microeconomic analysis is effectively used in describing the consumer's behavior. This model enables to explain intuitive tendency of consumers in decisions making when creating their optimal consumption strategy in market environment with changing parameters, in this type of problem it is usually price.

The paper points to certain features of market products basket, in case of analysis of the utilities industry. This specificity arises from the fact that usually consumers are not reasonably capable of product substitution of utilities industry, for example. gas, electricity and etc., with other goods with adequate performance and therefore is perceived as an exclusive product. This exclusivity can be formally expressed in the utility functions construction and in further consequent downstream analysis tasks.

Let's assume, that on relevant utilities industry market act $m$ consumers $S_i$ for $i = 1, 2, ..., m$. Goods, or services of utilities industries, having homogeneous nature, let's say distribution of electricity is provided by n subjects, suppliers Dj for j = 1, 2, …, n, while assuming, aggregate supply of products on the market is sufficiently high, to meet demand of every customers needs.

In below presented model is homogeneous product market considered therefore. Consumption decision variable $x_i$ is defined as the consumption of homogeneous products of utilities industry by $i$-th consumer $S_i$ and consumption of all other goods in the economy for the consumer is presented by calculated aggregate variable $x_{0i}$. If the utility from consumption of the products by consumer $S_i$ is given by the function $u_i(x_i)$, which reflects the level of utility in monetary units and calculated price of goods is normalized to value 1, than total utility of the consumer $S_i$ is expressed as a function $v_i(x_i, x_{0i})$ as follows

$$v_i(x_i, x_{0i}) = u_i(x_i) + x_{0i}$$
s. t.
$$v_i(x_i, x_{0i}) : R^2 \to R$$
$$u_i(x_i) : R \to R$$

Thus by such perceived utility function for further analysis can be interpreted as the overall utility of the "monetary units", which the consumer feels when purchasing $x_i$ units of the product of utilities industry and the simultaneous purchase of $x_{0i}$ units of aggregate basket of other goods that are highly standardized by price $p_n$=1 monetary unit.

Suppose further, utility function $u_i(x_i)$ is for every customer $S_i$ smooth, consequently. continuous and differentiable and its value is zero if commodity consumption is zero, so is $u_i(0) = 0$. Suppose further, marginal utility function $mu_i(x_i)$ is decreasing, so that the second derivative of utility function $u_i^{''}(x_i)$ relationship is

$$u_i^{''}(x_i) = \frac{dmu_i(x_i)}{dx_i} = \frac{d^2u_i(x_i)}{dx_i^2} < 0$$

and definition of the function of marginal utility $mu_i(x_i)$ and its functional value is

$$D(mu_i(x_i)) = \langle 0, \infty),$$
$$H(mu_i(x_i)) = (-\infty, \infty).$$

In other words, the marginal utility of goods studied is in the zone of utility growth positive, but its value gradually decreases to the maximum point of utility function where marginal utility value zero and further growth in consumption is a possible decline in marginal utility to the negative values. Utility $u_i(x_i)$ of purchasing $x_i$ units of goods then corresponds to the willingness of consumers to pay $x_i$ units by specific number of aggregate units of goods. Simply expressed, the consumer is willing to obtain $x_i$ units of goods of utilities industry to abandon $u_i(x_i)$ units of aggregate goods for a standardized unit price.

The behavior of the $i$-th consumer $S_i$ for each $i$ = 1, 2, ..., $m$ will be examined through the regular functions of optimization problems to maximize the total utility of $i$-th consumer $S_i$ subject to commodity specified consumption spending of consumer with limit $w$ and price of commodity $p$ which is for non-negative decision variables $x_i$ a $x_{0i}$ formulated as follows:.

$$v_i(x_i, x_{0i}) = u_i(x_i) + x_{0i} \quad \rightarrow \ \max$$

s. t.

$$px_i + x_{0i} = w_i$$

$$x_i, x_{0i} \geq 0$$

The above mentioned optimization problem of mathematical programming is the problem of maximization over set. Rewrite this problem to the standard form, i.e. minimization problem as the following:

$$-v_i(x_i, x_{0i}) = -u_i(x_i) - x_{0i} \quad \rightarrow \ \min \tag{1}$$

s.t.

$$px_i + x_{0i} = w_i \tag{2}$$

$$x_i, x_{0i} \geq 0 \tag{3}$$

## 2. Kuhn-Tucker conditions of optimality and its consequence

For optimization problem (1), (2), (3) is General Lagrange function formulated. Note only, general Lagrange function cover implicitly no negativity constraints of decision variables into Kuhn-Tucker conditions. General Lagrange function of mathematical programming problem (1), ..., (3) is formulated as follows:

$$\begin{aligned} L_i(x_i, x_{0i}, \lambda_i) = -v_i(x_i, x_{0i}) + \lambda_i(px_i + x_{0i} - w_i) = \\ -u_i(x_i) - x_{0i} + \lambda_i(px_i + x_{0i} - w_i) \end{aligned} \tag{4}$$

Kuhn-Tucker conditions of optimality for Lagrange function (4) *i*-th customer $S_i$ are formulated as follows:

$$\begin{array}{lll} \dfrac{\partial L_i(x_i, x_{0i}, \lambda_i)}{\partial x_i} \geq 0 & \dfrac{\partial L_i(x_i, x_{0i}, \lambda_i)}{\partial x_{0i}} \geq 0 & \dfrac{\partial L_i(x_i, x_{0i}, \lambda_i)}{\partial \lambda_i} = 0 \\ x_i \dfrac{\partial L_i(x_i, x_{0i}, \lambda_i)}{\partial x_i} = 0 & x_{0i} \dfrac{\partial L_i(x_i, x_{0i}, \lambda_i)}{\partial x_{0i}} = 0 & \\ x_i \geq 0 & x_{0i} \geq 0 & \end{array} \tag{5}$$

Kuhn-Tucker conditions of optimality (5) after substituting into analytical form of Lagrange function (4) and after following arrangement:

$$\begin{array}{lll} \dfrac{-\partial v_i(x_i, x_{0i})}{\partial x_i} + \lambda_i p \geq 0 & \dfrac{-\partial v_i(x_i, x_{0i})}{\partial x_{0i}} + \dfrac{\partial \lambda x_{0i}}{\partial x_{0i}} \geq 0 & \dfrac{\partial\left(-v_i(x_i, x_{0i}) + \lambda_i(px_i + x_{0i} - w_i)_i\right)}{\partial \lambda_i} = 0 \\ x_i\left(\dfrac{-\partial v_i(x_i, x_{0i})}{\partial x_i} + \lambda_i p\right) = 0 & x_{0i}\left(\dfrac{-\partial v_i(x_i, x_{0i})}{\partial x_{0i}} + \dfrac{\partial \lambda x_{0i}}{\partial x_{0i}}\right) = 0 & \\ x_i \geq 0 & x_{0i} \geq 0 & \end{array}$$

$$-\frac{\partial u_i(x_i)}{\partial x_i}+\lambda_i p \geq 0 \qquad -1+\lambda \geq 0 \qquad (px_i + x_{0i} - w_i) = 0$$

$$x_i\left(-\frac{\partial u_i(x_i)}{\partial x_i}+\lambda_i p\right)=0 \qquad x_{0i}(-1+\lambda)=0$$

$$x_i \geq 0 \qquad x_{0i} \geq 0$$

$$\begin{array}{llllll} -u_i^{'}(x_i)+\lambda_i p \geq 0 & (a) & -1+\lambda \geq 0 & (d) & px_i + x_{0i} = w_i & (g) \\ x_i(-u_i^{'}(x_i)+\lambda_i p)=0 & (b) & x_{0i}(-1+\lambda)=0 & (e) & & (6) \\ x_i \geq 0 & (c) & x_{0i} \geq 0 & (f) & & \end{array}$$

In other words, if the consumer decides for consumption strategy $(x_i, x_{0i})^*$, this means that consumption $x_i^*$ units of utilities industry for price $p$ and consumption $x_{0i}^*$ units of remaining goods from aggregate sectors with the unit price is maximizing its overall utility $v(x_i, x_{0i})^*$, so there must be such multiplier $\lambda^*$ for which the Kuhn – Tucker conditions (6) are met, so the vector of variables $(x_i, x_{0i}, \lambda)^*$ represents solution of the equations and inequalities system (a), (b) ... (g).

## Conclusion

The optimization model of customer behavior on utilities industry markets are investigated in this paper. From Kuhn-Tucker conditions of optimality are interpreted some business consequence

From constraint (g) is obvious, that vector of optimal consumption $(x_i, x_{0i})^*$ with price $p$ of products $x_i$ and unit price of aggregated sector $x_{0i}$ is possible to provide for customer budget $w$.

With this condition, then from the validity of conditions (b) shows that a positive aggregate consumption of goods and $x_0$ is positive for the volume of consumption of the product $x_i^*$, which maximizes utility. Necessarily true that in the peak point utility consumption of commodities is the marginal utility of consumption commodity equal to commodity price and the true

$$mu_i\left(x_i^*\right) = u_i^{'}\left(x_i^*\right) = \left[\frac{du(x)}{dx}\right]_{x=x_i^*} = p$$

This result further confirms the important theoretical postulate, namely that the consumer then increases consumption, in this case the product of utilities industries, while the marginal utility is not reaching the market price of the product.

This finding ultimately resulting indirectly from optimal conditions of Kuhn - Tucker (a), (b) where we see that, if the condition (a) implement the optimal structure of the vector $(x_i, x_{0i}, \lambda)^*$ as a sharp inequality:

$$-u_i^{'}(x_i) + p > 0$$

From conditions (b) arise that the consumer of utilities industry goods is not purchasing

$$x_i^* = 0$$

This finding is economically fully justified as this would increase the utility induced not cover the purchase of goods or its price.

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# PRICING MODELS IN REVENUE MANAGEMENT

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**Abstract**

Revenue management is the art and science of predicting real-time customer demand and optimizing the price and availability of products according to the demand. The paper is devoted to modeling of pricing in revenue management. The deterministic models assume that the seller has perfect information about the demand process. Deterministic models are easy to analyze and they provide a good approximation for the more realistic yet complicated stochastic models. Pricing policies with stochastic demand are more complex and harder to compute than with deterministic demand. Some approaches are presented in the paper.

**Keywords:** revenue management, dynamic pricing

## 1. Revenue management

Revenue Management (RM) is to sell the right product, to the right customer at the right time, for the right price through the right channel by maximizing revenue. RM is the art and science of predicting real-time customer demand and optimizing the price and availability of products according to the demand. RM addresses three basic categories of demand-management decisions:

- structural,
- price,
- and quantity decisions.

The RM area encompasses all work related to operational pricing and demand management. This includes traditional problems in the field, such as capacity allocation, overbooking and dynamic pricing, as well as newer areas, such as oligopoly models, negotiated pricing and auctions.

Recent years have seen great successes of revenue management, notably in the airline, hotel, and car rental business. Currently, an increasing number of industries is exploring to adopt similar concepts (see Talluri, van Ryzin, 2004). What is new about RM is not the demand-management decisions themselves but rather how these decisions are made. The true innovation of RM lies in the method of decision making.

Network revenue management models attempt to maximize revenue when customers buy bundles of multiple resources. The dependence among the resources in such cases is created by customer demand.

The basic model of the network revenue management problem is formulated as a stochastic dynamic programming problem whose exact solution is computationally intractable. Most approximation methods are based on one of two basic approaches: to use a simplified network model or to decompose the network problem into a collection of single-resource problems.

The Deterministic Linear Programming (DLP) method is a popular in practice. The DLP method is based on a bad assumption that demand is deterministic and static. Approximation methods based on extensions of the basic approaches are proposed

## 2. General model

The revenue management general model (Bitran, Caldentey, 2003) provides a global view of the different elements and their interrelations:

- Supply

- Product
- Information
- Demand

A seller has a fixed amount of initial capacity that is used to satisfy a price-sensitive demand during a certain selling period [0, $T$]. This initial capacity is modeled by an $m$-dimensional vector of $m$ resources. Capacity can be interpreted for example as rooms in a hotel, available seats for a specific origin-destination flight in a given day etc. Capacity is essentially given and the seller is committed exclusively to finding the best way to sell it. From a pricing perspective, two important attributes of the available capacity are its degree of flexibility and its perishability. Flexibility measures the ability to produce and offer different products using the initial capacity $C_0$. Perishability relates to the lack of ability to preserve capacity over time. As time progresses and resources are consumed, capacity decreases. The available capacity at time $t$ is denoted by $C_t = (c_1(t),\ldots,c_m(t))$.

A product is a sub-collection of the available resources. An $(m, n)$ matrix $A = [a_{ij}]$ is defined such that $a_{ij}$ represents the amount of resource $i$ used to produce one unit of product $j$. Every column $j$ of $A$ represents a different product and the collection $M = \{A_{.1}, \ldots, A_{.n}\}$ is the menu of products offered by the seller.

The knowledge of the system and its evolution over time is crucial to any dynamic pricing policy. Given an initial capacity $C_0$, a product menu $M$, and a demand and price processes, the observed history $H_t$ of the selling process is defined as the set of all relevant information available up to $t$. This history should include at least the observed demand process and available capacity, and it can also include some additional information such as demand forecasts.

The set of potential customers is divided into different segments each one having its own set of attributes. A $d$-dimensional stochastic process is defined $N(t, H_t) = (N_1(t, H_t), \ldots, N_d(t, H_t))$ where $N_j(t, H_t)$ is the cumulative potential demand up to time $t$ from segment $j$ given the available information $H_t$. An $(n, d)$ matrix $B(P) = [b_{ij}]$ is defined where $b_{ij}$ represents the units of product $i \in M$ requested by a customer in segment $j = 1, \ldots, d$. The demand depends on the pricing policy $P = \{p_t, t \in [0, T]\}$ where $p_t(i, H_t)$ is the price of product $i \in M$ at time $t$ given a current history $H_t$. The effective cumulative demand process in $[0, t]$ at the product level is defined as an $n$-dimensional vector $D(t, P, H) = B(P)N(t, H_t)$. The set of all admissible pricing policies, those that satisfy all the relevant constraints, is denoted by $\Pi$. The seller has the ability to partially serve demand if it is profitable to do so. An $n$-dimensional vector $S(t)$ that represents the cumulative sales up to time $t$ is defined.

The general revenue management problem is to find the solution to the following optimal control problem

$$\sup_{P,S} E_N \left[ \int_0^T p_t \, dS(t) \right]$$

subject to

$$C_t = C_0 - AS(t) \geq 0 \text{ for all } t \in [0, T],$$

$$0 \leq S(t) \leq D(t, P, H_t) \text{ for all } t \in [0, T],$$

$$P \in \Pi, \text{ and } S(t) \in H_t.$$

## 3. Deterministic models

The deterministic models assume that the seller has perfect information about the demand process. Deterministic models are easy to analyze and they provide a good approximation for the more realistic yet complicated stochastic models. Deterministic solutions are in some cases asymptotically optimal for the stochastic demand problem (Cooper, 2002).

The simplest deterministic model considers the case of a monopolist selling a single product to a price sensitive demand during a period [0, *T*]. The initial inventory is $C_0$, demand is deterministic with time dependent and price sensitive intensity $\mu(p, t)$. The instantaneous revenue function $r(p, t) = p\mu(p, t)$ is assumed to be concave as in most real situations. The general revenue management problem can be written in this case as follows.

$$\max_p \int_0^T p_t \mu(p_t, t)dt$$

subject to (1)

$$\int_0^T \mu(p_t, t)dt \le C_0 .$$

This is a standard problem in calculus of variations. The optimality condition is given by

$$p_t^* = \lambda - \frac{\mu(p_t^*, t)}{\mu_p(p_t^*, t)},$$

where $\lambda$ is the Lagrangian multiplier for the constraint, $\mu_p$ is the partial derivative of $\mu$ with respect to the price.

For the case of a time homogeneous demand intensity ($\mu(p, t) = \mu(p)$) a fixed price solution can be shown to be optimal over the entire selling period [0, *T*].

Let $p^* = \text{argmax } \{p\mu(p) : p \geq 0\}$ be the price policy that maximizes the revenue rate and $\mu^* = \mu(p^*)$ be the corresponding demand intensity. Similarly, let $p^0$ be the solution to $\mu(p^0)\ T = C_0$ and $\mu^0 = \mu(p^0)$ be the corresponding demand intensity.

The single product revenue management problem (1) with homogenous demand intensity $\mu(p)$, and concave revenue rate $r(p) = p\mu(p)$ has solution:

- if $\mu^* T \leq C_0$ then the optimal price is $p^*$ and the optimal revenue is equal to $p^*\mu^*T$.
- if $\mu^* T > C_0$ then the optimal price is $p^0$ and the optimal revenue is equal to $p^0\ C_0$.

This result is used as a building block for constructing heuristics and bounds for the stochastic models.

## 4. Stochastic models

Pricing policies with stochastic demand are more complex and harder to compute than with deterministic demand. On the other hand, stochastic models are clearly used more appropriately to describe real life situations where the paths of demand and inventory are unpredictable over time and managers are forced to react dynamically by adjusting prices as uncertainty reveals itself. The natural way to tackle a problem of this type is by using stochastic dynamic programming techniques. At every decision point during the selling season, the manager collects all relevant information about the current inventory positions and sales and establishes the prices at which the products should be sold. Most of the research has been done for the single product case under Markovian assumptions on the demand process. In this setting, the inventory levels are the only relevant information that managers need to make pricing decisions.

In the single product case, the initial capacity $C_0$ is a scalar representing the number of units of the product that are available at time $t = 0$. The value function $V_t(C_t)$ is defined at time $t$ if the inventory is $C_t$, that is, $V_t(C_t)$ is the optimal expected revenue from time $t$ to the end of the season given that the current inventory position at time $t$ is $C_t$. Time $t$ has been modeled in the literature as either a continuous or discrete variable. From a practical perspective, managers will revise their price decisions only at discrete points in times. However, the explosive growth of the Internet and E-commerce make the continuous time model much more suitable for practical uses.

**Static model**

The simplest approach to the problem is the static price solution. The pricing policy is restricted to be a fixed price during the entire season, *i.e.*, $p_t = p$ for all $t \in [0, T]$. This type of static policy is appropriate for products having one or more of the following characteristics:

- short selling period,
- high costs of changing prices,
- legal regulations that force the price to be fixed.

The fixed price model is simple and easy to implement and control. Hence, even if price changes are possible, managers often choose to use the static fixed price approach. The fixed price model is asymptotically optimal in some situations. In this single product fixed price model is given by

$$V(C_0, T) = \max_{p \geq 0} V(C_0, p, T) = \max_{p \geq 0} E[p \min\{D(p, T); C_0\}]$$

where $D(p, T)$ is the random variable representing the cumulative demand in $[0, T]$ at a price $p$. Closed-form solutions for this problem are not available for the general case of an arbitrary distribution of $D(p, T)$. The optimal price can be characterized in terms of the demand elasticity. Let $f(D, p, T)$ be the probability mass function of $D(p, T)$. The demand elasticity with respect to price is defined as

$$e = \frac{p f_p(D, p, T)}{f(D, p, T)}.$$

where $f_p(D, p, T)$ is the partial derivative of $f(D, p, T)$ with respect to $p$. The first order optimality condition for the solution of is given by

$$\frac{E[\min\{D; C_0\} e(D, p, T)]}{E[\min\{D; C_0\}]} = -1.$$

The weighted expected value of the elasticity has to be equal to -1, where the weight is given by the level of sales $\min\{D; C_0\}$.

**Dynamic model**

The dynamic price model is a continuous time model where demand follows a Poisson process with fixed intensity $\mu$. An arriving customer at time $t$ has a reservation price $r_t$ for the product. From the seller perspective, the reservation price $r_t$ is a random variable with distribution $F(r; t)$. Two cases can be considered (Kincaid and Darling, 1963). In the first case, the seller does not post prices but receives offers from potential incoming buyers, which he either accepts or rejects. It is assumed that arriving customers offer their reservation price $r_t$. In the second case, the seller posts the price $p_t$ and arriving customers purchase the product only if $p_t \le r_t$. The demand process in this situation is Poisson with intensity $\mu(1 - F(p_t, t))$. Optimality conditions for the value function $V_t(Ct)$ and the optimal price $p_t(C_t)$ are derived for both cases

$$p_t(C_t) = \frac{1 - F(p_t(C_t), t)}{f(p_t(C_t), t)} + V_t(C_t) - V_t(C_t - 1).$$

The problem of computing an optimal price strategy reduces to the computation of the opportunity cost $V_t(C_t) - V_t(C_t - 1)$. In general, there is no exact closed-form solution for the optimal price strategy. It can be shown that

$$\frac{V_t^{\det}(C_t)}{V_t(C_t)} \ge 1 - \frac{1}{2\sqrt{\mu(p^{\det})(T-t)}},$$

where $\mu(p^{\det})$ is the demand intensity at price $p^{\det}$. The deterministic price heuristic is asymptotically optimal as $T$ goes to infinity.

## 5. Conclusions

Pricing in revenue management is a complex problem. Some approaches solve the problem in simplified forms. There are deterministic and stochastic models. Deterministic models are easy to analyze and they provide a good approximation for the more realistic yet complicated stochastic models. Stochastic models are clearly used more appropriately to describe real life situations. Static or dynamic approaches can be used.

## Acknowledgements

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# AN EXAMPLE OF A BENEFICIAL CARTEL ON BOTH SIDES OF A MARKET

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**Abstract:** In this paper we give an example of a cartel that contains firms on both sides of a market and improves consumer welfare. There is the monopsonistic buyer, who is a retailer, and four firms producing two goods that are substitutes. We compare a non-collusive outcome with a collusive one. In the former all producers are price takers and the monopsonist takes the inverse supply functions into account in his purchasing decision. In the latter the firms maximize the sum of their profits. The produced quantities are higher and the prices for consumers are lower in the collusive outcome.

## INTRODUCTION

A general textbook knowledge maintains that cartels are bad for consumers because they reduce produced quantities and increase prices. (See, for example, Hirshleifer, 1988, p. 251-256.) This view is held also by officials of competition authorities. For example, according to the chairwoman of the Antimonopoly Office of the Slovak Republic, "Cartels in general bring no positive effects." (Paroulková, 2008, p. 2).

This textbook view assumes that a cartel contains only firms on the supply side of a market. It is also based on an assumption that products of cartel members are either identical or substitutes. (In competition policy considerations these assumptions are often only implicit.) In the present paper we show, by a simple example, that a cartel containing firms on both sides of a market can increase welfare of final consumers. That is, it can increase outputs and decrease prices.

(We keep the assumption that outputs of the producers in a cartel are either identical or they are substitutes.)

We work with a wholesale market and a retail market in our example. In the former there are four producers and the monopsonistic buyer. Two types of good, which are substitutes, are sold in this market. Each of them can be produced by two firms. The monopsonist sells the purchased goods in the retail market, in which he is the monopolist. We compare a non-collusive outcome with a collusive one. In the former all producers are price takers and the monopsonist takes the inverse supply functions into account in his decision on purchased quantities. In the latter the firms maximize the sum of their profits.

Stability of a cartel outcome is outside the scope of the paper. Nevertheless, it can be shown that, even without a binding agreement and from the static (i.e. one period) perspective, it is immune to deviations by all coalitions (including the grand coalition and singleton coalitions). Namely we can construct a strategic form non-cooperative game, in which the described collusive outcome is a strong Nash equilibrium. (The concept of a ‘strong Nash equilibrium' was developed by Robert Aumann, 1959. Despite the title of his paper it is a solution concept for non-cooperative games. See also Bernheim et al.1987, p. 2-3.)

The paper is organized as follows. In the following section we describe the wholesale market and the retail market used in our example. The second section deals with the non-collusive benchmark, in which producers are price takers. In the third section we study the collusive outcome based on the maximization of the sum of firms' profits. The concluding section contains some economic policy implications.

## 1. THE ANALYZED MARKETS

There are five firms in the wholesale market. $J = \{1,2,3,4\}$ is the set of producers. They produce two types of good that are substitutes. Firms 1 and 2 produce the first type of good. Firms 3 and 4 produce the second type of good. For each producer $j \in J$ his capacity allows him to produce at most 5 units of the good and his cost function $c_j : [0,5] \rightarrow [260]$ has the form $c_j(y_j) = 10y_j^2 + 10$. Firm 5 is a retailer. It sells the goods purchased from firms belonging to $J$ in the retail market, in which it is the monopolist. Producers' capacities generate also upper bounds on quantities of goods that can be sold in the retail market.

We denote by $Q_I$ the aggregate output of the first type of good and by $Q_{II}$ the aggregate output of the second type of good. These aggregate outputs equal the quantities sold in the retail market.

Firm 5' costs of selling goods in the retail market are captured by its cost function $c_5 : [0,10]^2 \to [0,35]$ that has the form $c_5(Q_I, Q_{II}) = Q_I + Q_{II} + 15$. These costs can include, for example, labor, transportation, and handling costs. Their fixed component contains depreciation for equipment of shop(s). The costs captured by function $c_5$ do not include expenditures on purchase of goods from firms 1 and 2.

The inverse demand functions for the retail market, $P_I : [0,10]^2 \to [0,121]$ and $P_{II} : [0,10]^2 \to [0,121]$, have the form

$$P_I(Q_I, Q_{II}) = \max\{121 - 10Q_I - 5Q_{II}, 0\} \qquad (1)$$

and

$$P_{II}(Q_I, Q_{II}) = \max\{121 - 5Q_I - 10Q_{II}, 0\}. \qquad (2)$$

## 2. THE NON-COLLUSIVE BENCHMARK

The real world is the world of imperfect competition. A benchmark for assessment of the impact of collusion on consumer welfare should reflect this. Therefore, we use a non-collusive benchmark in which the monopsonist exercises his monopsony power in the wholesale market. Following the tradition in microeconomics literature, which treats the monopolist as a quantity choosing firm, we let the monopsonist in the wholesale market (who is also the monopolist in the retail market) to choose the quantities of both types of goods that he wants to purchase (and then sell in the retail market). Of course, in doing so he maximizes his profit and (as he has monopsony power in the wholesale market) he takes into account the effect of his quantity decisions on prices he has to pay. These prices are determined by inverse supply functions for the wholesale market. The latter functions are derived under the assumption of price taking behavior of producers.

In this example it is enough to restrict attention to prices in the wholesale market that do not exceed 100 financial units. (For higher prices the competitive supply would be determined not by the equality of price and marginal costs but by the capacity constraint. Functional values of inverse supply functions cannot exceed 100 financial units.) Then the short run supply function of each producer $j \in J$ , $s_j : [0,100] \to [0,5]$ (which is derived from the maximization of his profit), has the form $s_j(p_j) = 0.05 p_j$ . Thus, the short run market (aggregate) supply function for type $k \in \{I, II\}$ of good, $S_k : [0,100] \to [0,10]$, has the form $S_k(p_k) = 0.1 p_k$ . This gives the short run inverse market supply function for type $k \in \{I, II\}$ of good, $S_k^{-1} : [0,10] \to [0,100]$, in the form $S_k^{-1}(Q_k) = 10 Q_k$ .

The quantities purchased by the monopsonist are given by the solution of the maximization program

$$\max(121-10Q_1-5Q_2)Q_1+(121-5Q_1-10Q_2)Q_2-10Q_1^2-10Q_2^2-Q_1-Q_2-15 \quad (3)$$

$$subject\ to\ Q_1 \in [0,10],\ Q_2 \in [0,10],\ 10Q_1+5Q_2 \le 121,\ 5Q_1+10Q_2 \le 121. \quad (4)$$

The first two constraints ensure that output of neither type of good exceeds aggregate capacity for its production. The third (fourth) constraint ensures that the price of the first (second) type of good is given by the first expression in the composite brackets on the right hand side of (1) ((2)). (The profit maximizing monopsonist will not behave in the way violating the third or fourth constraint. Therefore, these constraints cannot cut off an optimal solution of the optimization program maximizing the monopsonist's profit.)

The solution of the maximization program (3)-(4) is $\hat{Q}_1 = 2.4$ and $\hat{Q}_2 = 2.4$. This gives prices $S_I^{-1}(2.4,2.4) = S_2^{-1}(2.4.,2.4) = 24$ in the wholesale market (i.e. each producer sells his output for the unit price equal to 24 financial units), output $\hat{y}_j = 1.2$ for each producer $j \in J$, prices $P_I(2.4,2.4) = P_{II}(2.4,2.4) = 85$ in the retail market, and profits $\hat{v}_j = 4.4$ for each $j \in J$ and $\hat{v}_5 = 273$.

Since producers' profits are positive, at price equal to 24 financial units functional values of their short run supply functions coincide with functional values of their long run supply functions. Therefore, the non-collusive equilibrium computed above does not change unless a new firm enters either the wholesale market or the retail market.

## 3. THE COLLUSIVE OUTCOME

We want to compare the non-collusive equilibrium from the preceding section with a collusive outcome that strictly Pareto dominates it in terms of firms' profits. Clearly, in such collusive outcome each producer has to be active (i.e. his output has to be positive). First we compute outputs that maximize the sum of firms' profits. Taking into account producers' cost functions, when all of them are active, the minimization of total production costs (which is necessary for the maximization of the sum of firms' profits) requires that aggregate output of each type of good is divided equally between his two producers. Thus, aggregate outputs maximizing the sum of firms' profits are given by the solution of the maximization program

$$\max(121-10Q_1-5Q_2)Q_1+(121-5Q_1-10Q_2)Q_2-Q_1-Q_2-5Q_1^2-5Q_2^2-55 \qquad (5)$$

*subject to* (4).

(Clearly, the outputs maximizing the sum of firms' profits cannot violate the last two constraints in (4).)

The solution of the maximization program (5)-(4) is $Q_1^* = 3$ and $Q_2^* = 3$. This gives sum of firms' profits equal to 305 financial units, output $y_j^* = 1.5$ for each producer $j \in J$, and prices $P_I(3,3) = P_{II}(3,3,) = 76$ in the retail market. When each producer sells his output to the monopsonist for price equal to 25 financial units, firms' profits are $v_j^* = 5$ for each $j \in J$ and $v_5^* = 285$. Since $v_k^* > \hat{v}_k$ for each $j \in J \cup \{5\}$, it is in the interest of all firms to switch from the non-collusive equilibrium described in the preceding section to the collusive outcome described here and stick to the latter unless a new firm enters either the wholesale market or the retail market.

Switching from the non-collusive equilibrium to the collusive outcome increases output and decreases price of each type of good. Thus, it clearly increases consumer welfare. We give a brief intuitive explanation why increase in each producer's output enables increase in each firm's profit. In the non-collusive setting from Section 2 an increase in the monopsonist's purchase of type $k \in \{I, II\}$ of good increases his expenditures on it given by the product of the purchased quantity and the functional value of the inverse supply function for it (for a 'small' increase, starting from the original purchased quantity $Q_k$, the expenditures increase by $S_k^{-1}(Q_k) + Q_k dS_k^{-1}(Q_k)/dQ_k > S_k^{-1}(Q_k)$, where the term on the right hand side equals the marginal costs of each producer of type $k$ of good at the original purchased quantity). The collusive arrangement makes possible an increase in the monopsonist's purchase of type $k \in \{I, II\}$ of good with the increase in his expenditures on it that are only slightly higher than the compensation of producers' increased costs (for a 'small' increase the compensation of producers' increased costs equals to their marginal costs at their original outputs).

It is worth noting that the collusive arrangement described in this section maximizes the sum of firms' profits also when a firm has an option to exit the wholesale market and by doing so avoid fixed costs.

## CONCLUSIONS

Our example resembles relationship between a chain-store and its suppliers. Chain-stores have considerable market power in their input markets (that can be approximated in a model by their monopsony power) and also considerable market power in local retail markets. The present paper suggests that (for some values of cost and demand functions) allowing collusion in chain-stores' input markets could not only improve the situation of their suppliers (while increasing further profits of chain-stores) but also make consumers better off.

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# ANALYSIS OF THE MUTUAL RELATIONSHIPS BETWEEN THE EXCHANGE RATES AND THE STOCK INDICES

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**Abstract:** The main aim of this paper is to analyze the relationships between the exchange rates and the stock indices. The analysis was done for the Visegrad countries (the Czech Republic, Hungary, Poland and Slovakia) using the daily data from January 1, 1999 to May 21, 2010. We applied the Phillips-Perron unit root test to examine the existence of unit roots. Both time series of almost all countries contained one unit root (the only exception was Poland); the existence of the long-run relationships between these time series in individual countries was tested based on Johansen cointegration procedure and was not confirmed. Since one of the typical features of the financial time series is volatility clustering, we applied the ARCH methodology to capture it and carried out the Granger causality tests both on original return series and volatility-filtered series with different results.

**Keywords:** unit root test, cointegration test, Granger causality concept, impulse responses analysis, GARCH model, volatility-filtered series

## 1. Introduction

The analysis of the mutual relationships between the exchange rates and the stock indices has been very popular for a long time. In the literature there are two main approaches dealing with this theme (see e.g. [11], [13], [14]). The goods market approach concentrates on the current account of the balance of payment and states that exchange rates would lead stock prices. According to the portfolio balance approach the stock price fluctuations influence the exchange rates movements. There have been many studies (see e.g. [3], [9], [11], [13], [14]) done in this area using various procedures (Granger causality, impulse responses analysis, variance decompositions, cointegration procedures, autoregressive conditional heteroskedastic – ARCH methodology, etc.), various countries, various data span and periodicity. The presented results and conclusions about validity of the above mentioned approaches sharply differ depending inter alia on the specific characteristics of the country analyzed.

The aim of this paper is to analyze the relationships between the exchange rates of the individual V4 countries'[1] national currencies against the US dollar and the stock index of the corresponding V4 country using the daily data and to examine the impact of the volatility effects on the results of the Granger causality tests.

## 2. Data and Methodology

To analyze the bivariate relationships we used the unit root test, cointegration test, Granger causality concept, impulse responses analysis and modifications of ARCH models. The whole analysis was done in econometrical software EViews 5.1.

[1] V4 countries = Visegrad countries (the Czech Republic, Hungary, Poland, Slovakia)

### 2.1. Data

This paper used daily data from January 1, 1999 to May 21, 2010[2] with exclusion of days in which the exchange rate or the stock index in corresponding country was not defined. The analyzed exchange rates and stock indices were received from web pages [15] and [16]. In case of the Czech Republic we analyzed the relationship between the exchange rate CZK/USD and the stock index PX (2860 observations), in case of Hungary the analyzed time series were HUF/USD and BUX (2841 observations). For Poland we used the exchange rate PLN/USD and the stock index WIG (2855 observations). Time series SKK/USD and SAX were analyzed in case of Slovakia (2261 observations). One of the basic features of the financial time series is the non-stationarity. The non-stationarity in variance can be solved e.g. by using the logarithmic transformation of the corresponding time series. Since the existence of the non-stationarity in mean also represents a serious problem in econometric analysis, it is necessary to deal with testing of it using e.g. various unit root tests[3].

### 2.2. Phillips – Perron unit root test

All the above mentioned time series were tested for the existence of the unit root using the Phillips – Perron (PP) test, since we can expect that there are autocorrelation and ARCH effects in the series and this test is robust to strong autocorrelation and heteroskedasticity (see e.g. [4], [11]). The results of the PP test for logarithmic transformations of the individual time series are summarized in table 1. From the results it is clear that all the analyzed time series (with exception of *lwig* which is stationary) had one unit root,

[2] In case of Slovakia the data span was (as a result of the euro adoption) shortened to the end of 2008.
[3] For more information about various unit root tests, their advantages and disadvantages see e.g. [1], [4], [7].

i.e. are non-stationary I(1). The results of this test will be taken into account for further analysis.

Table 1 Results of the PP unit root test

| | | Czech Republic | | Hungary | | Poland | | Slovakia | |
|---|---|---|---|---|---|---|---|---|---|
| | | *lczkusd* | *lpx* | *lhufusd* | *lbux* | *lplnusd* | *lwig* | *lskkusd* | *lsax* |
| Level | trend & intercept | -2.99 | -1.09 | -2.34 | -1.62 | -2.54 | -5.95 *** | -2.65 | -2.06 |
| | intercept | -0.6 | -1.24 | -1.49 | -1.02 | -1.29 | - | - | -0.11 |
| | without both trend & intercept | -0.87 | 1.12 | 0.04 | 1.20 | -0.28 | - | - | 2.53 |
| if. | trend & intercept | -51.20 *** | -50.31 *** | -53.89 *** | -50.05 *** | -53.25 *** | - | -46.58 *** | -47.15 *** |
| Conclusion | | I(1) | I(1) | I(1) | I(1) | I(1) | I(0) | I(1) | I(1) |

*Note: The symbol *** denotes the rejection of the null hypothesis at the 0.01 significance level.*

### 2.3. Johansen cointegration procedure

The use of the non-stationary variables can lead to the spurious regression. In order to solve this problem we can difference the series until the stationarity is achieved and to use the differenced series for analysis. In this context it is necessary to mention that the differencing process eliminates the useful long-run information about relationships among variables. In case of non-stationary variables it is suitable to use the conception of cointegration. There are two widely used cointegration procedures – the Engle – Granger procedure and the Johansen procedure (see e.g. [1], [4], [6], [10]). We employ the Johansen procedure [10] based on maximum likelihood method in order to examine the existence of cointegration between following groups of time series: *lczkusd* and *lpx*, *lhufusd* and *lbux*, *lskkusd* and *lsax*. The cointegration between time series *lplnusd* and *lwig* doesn't exist because of different order of integration of the series. The results of the cointegration test using both the $\lambda_{trace}$ and $\lambda_{\max}$ statistics are in table 2[4]. The results in table 2 show that there exist no cointegration relationship (i.e. no long-run relationship) between the analyzed time series. In order to test the short-run relationships we will further apply the Granger causality test and the impulse responses analysis.

Table 2 Results of Johansen cointegration procedure

| Cointegration between | Lags in VAR | r | N – r | $\lambda_{trace}$ | $\lambda_{\max}$ |
|---|---|---|---|---|---|
| *lczkusd* | 3 | 0 | 2 | 5.47 (15.49) | 4.74 (14.26) |
| *lpx* | | 1 | 1 | 0.73 (3.84) | 0.73 (3.84) |
| *lhufusd* | 3 | 0 | 2 | 6.10 (15.49) | 5.24 (14.26) |
| *lbux* | | 1 | 1 | 0.86 (3.84) | 0.83 (3.84) |
| *lskkusd* | 1 | 0 | 2 | 5.41 (15.49) | 5.22 (14.26) |
| *lsax* | | 1 | 1 | 0.19 (3.84) | 0.19 (3.84) |

*Note: Critical MacKinnon – Haug – Michelis values for significance level 0.05 are in parenthesis.*

### 2.4. Granger causality test

Granger causality test is a useful tool for analysis of the relationship between the time series. Using the Granger causality concept, we can say, that the time series $x_t$ Granger-causes time series $y_t$ if $y_t$ can be predicted better by using past values of $x_t$ than by using only the historical values of the $y_t$. If this doesn't hold, we can say that $x_t$ doesn't Granger-cause $y_t$. The situation whether the time series $y_t$ Granger-causes the time series $x_t$

[4] The number of lags of the individual Vector Autoregression (VAR) models was determined using the Schwarz information criterion (SC). In case of serial correlation the number of lags was appropriately increased. In the next step it is necessary to decide about the inclusion of the deterministic components (constant and/or trend) into the cointegrating equation (CE) and VAR model. The existence of cointegration was tested for all possible variants, but the results were in all cases the same. The results presented in table 2 are for model with constant in CE and VAR. The symbol *N* denotes the number of variables and $r$ is the number of cointegrating vectors.

can be tested in analogical way. Since the analyzed time series were non-stationary (with exception of *lwig*) and there was no cointegration between them, the Granger causality test had to be applied on first differences of them without inclusion of the error correction terms[5]. The results of the Granger causality test are in table 3[6]. Taking into account the calculated values of the Wald F-statistics and corresponding p-values we can conclude that at the 0.05 significance level the bilateral causality was confirmed in case of the Czech Republic and Hungary and unidirectional causality from exchange rate returns to stock returns in case of Slovakia. Slightly different are the results when we take into account the 0.01 significance level. In such case we can speak about the bilateral causality only in case of Hungary, the unidirectional causality from exchange rate returns to stock returns in the Czech Republic and independence in case of Slovakia.

Since the F – test results don't reveal whether changes in the value of a given variable have a positive or negative effect on other variable(s) in the system, or how long it would take for the effect of that variable to die away, the VAR's impulse responses (IR) were analyzed. [3]

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[5] For more information about the Granger causality test see e.g. [3], [13], [17].
[6] The number of lags was determined using the SC and in case of serial correlation was the number of lags appropriately increased.

Table 3 Granger causality tests

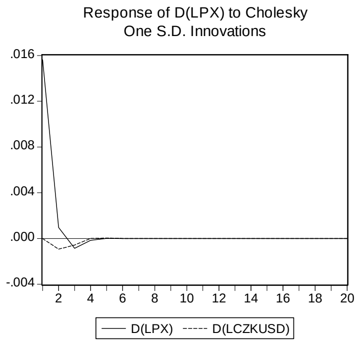
| $H_0$ | Lags | F - statistics | p-value |
|---|---|---|---|
| $d(lczkusd) -/\rightarrow d(lpx)$<br>$d(lpx) -/\rightarrow d(lczkusd)$ | 2 | 6.54<br>3.29 | 0.0015<br>0.0375 |
| $d(lhufusd) -/\rightarrow d(lbux)$<br>$d(lbux) -/\rightarrow d(lhufusd)$ | 12 | 2.20<br>2.90 | 0.0096<br>0.0005 |
| $d(lskkusd) -/\rightarrow d(lsax)$<br>$d(lsax) -/\rightarrow d(lskkusd)$ | 2 | 3.10<br>0.01 | 0.0452<br>0.9909 |

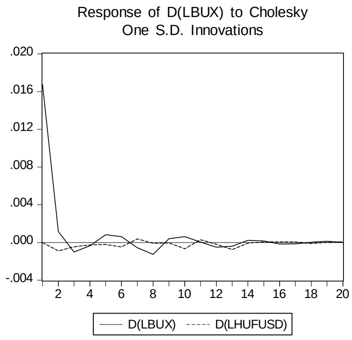
*Note: Symbol* $-/\rightarrow$ *implies "doesn't Granger-cause".*

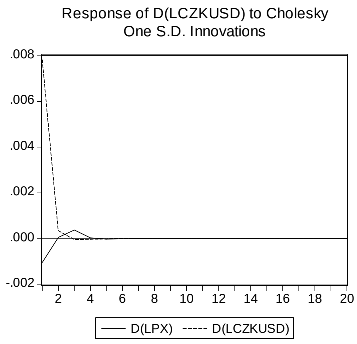
### 2.5. Impulse responses analysis

The IR analysis was also used to examine the short-run dynamic relations between stock returns and exchange rate returns. The essence of the IR analysis is based on the fact that for each variable from each equation separately, a unit shock is applied to the error, and the effects upon the VAR system over time are recorded. The IR functions from shocks of each variable (stock return, exchange rate return) using the Cholesky decomposition (with d.f. adjustment) are shown on figure 1.

According to the figure 1 we can state that there is no discernible reaction of the stock returns to one standard deviation innovation in exchange rate returns and vice versa, but the reaction of exchange rate (stock) returns to the innovation in itself is at the beginning of the tested period quite sharp (2 to 3 days). In case of Hungary (VAR with 12 lags) is the transmission of the effect of the shock to exchange rate (stock) returns to the stock (exchange rate) returns visible for about 14 days.


Response of D(LPX) to Cholesky
One S.D. Innovations
D(LPX) D(LCZKUSD)


Response of D(LBUX) to Cholesky
One S.D. Innovations
D(LBUX) D(LHUFUSD)


Response of D(LCZKUSD) to Cholesky
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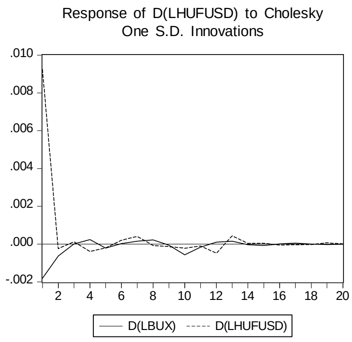
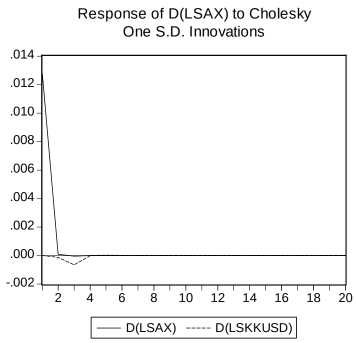
D(LPX) D(LCZKUSD)


Response of D(LHUFUSD) to Cholesky
One S.D. Innovations
D(LBUX) D(LHUFUSD)


Response of D(LSAX) to Cholesky
One S.D. Innovations
D(LSAX) D(LSKKUSD)

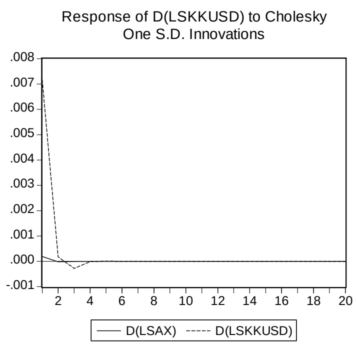

Response of D(LSKKUSD) to Cholesky
One S.D. Innovations
D(LSAX) D(LSKKUSD)

Figure 1 Impulse responses of variables to Cholesky one standard deviation innovations

### 2.6. Volatility effects

Since there have been many papers published in the area of exchange rate returns and stock returns interactions, only recently have attempts been made to analyze the impact of the volatility effects. In order to capture the volatility in return series the ARCH methodology was used [5]. We used the generalized ARCH (GARCH) models [2] for stock returns and exchange rate returns of the form $dlx_t = c + \varepsilon_t$; $h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}$,

where $dlx_t$ denotes the logarithmic return series $x_t$, $\varepsilon_t$ is a shock term, $h_t$ is the conditional variance, $p$ is the order of a GARCH term and $q$ is the order of an ARCH term and $c, \alpha_0, \alpha_i, \beta_i$ are the unknown parameters. Table 4 presents the results for the GARCH models in case of individual exchange rate returns and stock returns and also the results of the standardized residual tests.

The estimated parameters for ARCH and GARCH terms were in all cases statistically significant at the significance level 0.01, which means that the use of these models was adequate. Based on the Ljung – Box Q-statistics we can see that the standardized residuals and also the squared standardized residuals from the above mentioned models are uncorrelated till the lag 200 (at the significance level 0.05 and 0.01) and the ARCH LM test confirmed the absence of ARCH effects. The results of the Jarque – Bera tests show the violation of the condition of normal distribution and therefore the estimations are consistent only as quasi maximum likelihood. Since the existence of volatility effects was confirmed, these can influence the results of the Granger causality tests (see e.g. [13]). We also apply the Granger causality tests on volatility filtered series.

Table 4 Estimation of the GARCH models[7]

| | Czech Republic | | Hungary | | Slovakia | |
|---|---|---|---|---|---|---|
| | *d(lczkusd)* | *d(lpx)* | *d(lhufusd)* | *d(lbux)* | *d(lskkusd)* | *d(lsax)* |
| *c* | $-3.10^{-4}$** | 0.001*** | $-6.10^{-5}$ | $8.10^{-4}$*** | $-4.10^{-4}$** | 0.001*** |
| $\alpha_0$ | $4.10^{-7}$*** | $5.10^{-6}$*** | $9.10^{-7}$*** | $6.10^{-6}$*** | $7.10^{-7}$*** | $3.10^{-8}$*** |
| $\alpha_1$ | 0.035*** | 0.13*** | 0.065*** | 0.09*** | 0.02*** | 0.05*** |
| $\alpha_2$ | - | - | - | - | - | -0.05*** |
| $\beta_1$ | 0.96*** | 0.85*** | 0.93*** | 0.89*** | 0.97*** | 1.70*** |
| $\beta_2$ | - | - | - | - | - | -0.70*** |
| *Q(200)* | 161.47 | 225.90* | 185.10 | 162.11 | 228.28* | 197.04 |
| $Q^2(200)$ | 188.39 | 188.34 | 200.15 | 151.78 | 149.34 | 135.68 |
| *LM(1)* | 0.08 | 0.98 | 1.27 | 0.95 | 0.13 | 0.002 |
| *J-B* | 525.65*** | 268.19*** | 145.56*** | 109.81*** | 266.80*** | 3413.04*** |

*Note: The symbols ***, **, * denote the rejection of the null hypothesis at the 0.01, 0.05 and 0.1 significance level.*

### 2.7. Granger causality test for volatility – filtered series

The volatility-filtered series (i.e. residuals divided by the predicted value of volatility, standardized residuals) were used to test the Granger causality. The results of the Granger causality tests together with the Wald F-statistics and corresponding p-value are in table 5 from which it seems to be clear, that the Granger causality for volatility-filtered series was confirmed at the 0.05 significance level only in case of Hungary (stock returns $\rightarrow$ exchange rate returns), but at the 0.01 significance level in no case. The same results (no Granger causality) were achieved at the 0.01 significance level till the

[7] In case of *d(lpx)* it was necessary to include an AR(1) term into the mean equation and in case of *d(lbux)* a MA(1) term.

lag 12 in all analyzed cases. The same results were confirmed also by the impulse responses analysis.[8]

Table 5 Granger causality tests for volatility-filtered series

| $H_0$ | Lags | F - statistics | p-value |
|---|---|---|---|
| $d(lczkusd) -/\rightarrow d(lpx)$<br>$d(lpx) -/\rightarrow d(lczkusd)$ | 1 | 2.56<br>0.01 | 0.1097<br>0.9079 |
| $d(lhufusd) -/\rightarrow d(lbux)$<br>$d(lbux) -/\rightarrow d(lhufusd)$ | 1 | 2.50<br>3.99 | 0.1138<br>0.0459 |
| $d(lskkusd) -/\rightarrow d(lsax)$<br>$d(lsax) -/\rightarrow d(lskkusd)$ | 1 | 0.06<br>0.04 | 0.8135<br>0.8376 |

*Note: Symbol* $-/\rightarrow$ *implies "doesn't Granger-cause".*

## 3. Conclusion

This paper investigates the relationships between the exchange rates and the stock indices. The analysis based on Johansen cointegration procedure has shown that there is no long-run relationship between these two variables. The analysis was not done for Poland because of different order of integration of the used variables. Using the Granger causality concept both the goods market approach and the portfolio balance approach were confirmed in case of Hungary and goods market approach in case of the Czech Republic based on the 0.01 significance level. Since the ARCH terms were present in both return series in all three countries (the Czech Republic, Hungary and Slovakia), the Granger causality tests were applied also on

[8] The graphs are not published in order to save the space, but can be provided by the author upon request.

volatility-filtered series. The analysis showed only the existence of unidirectional causality in coincidence with the portfolio balance approach in case of Hungary using the 0.05 significance level and independence for all countries using the 0.01 significance level. It means that the volatility effects had an influence on Granger causality test results.

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# MULTICRITERIA EVALUATION OF INVESTMENTS INTO RENEWABLE ENERGY SOURCES

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**Abstract:** The paper focuses on evaluation of five renewable energy sources with respect to sixteen criteria divided into five main groups which reflect technical, economic, social, environmental, and strategic aspects of the investment. For this purpose the Analytic Hierarchy Process (AHP) was used as one of the most often used tools for solving multiple criteria decision making (MCDM) problems. In this paper renewable energy sources are also evaluated by other MCDM methods, such as WSA, ELECTRE I, PROMETHEE, and TOPSIS. The analysis is based on the data set which describes the situation in investments into renewable energy sources in the Czech Republic. The advantages and disadvantages of different approaches are discussed in the end of the paper as well as their different results.

**Keywords:** renewable energy sources, multicriteria decision making, AHP

## 1. Introduction

Many of recent studies concluded increasing living standard in many countries over the years The necessary consequence of this fact is the increase in energy use. Energy demand was covered by the use of fossil fuels (e.g. coal, petroleum, etc.) but their excessive use led to decrease of these sources. In addition, we cannot ignore the environmental impact from their use. These are reasons for effort to use and develop more environmentally friendly forms of energy. There are five real possibilities of renewable energy sources in the Czech Republic. The first ones are wind power stations for wind energy production. The second possibility is a photovoltaic power station based on obtaining of sun energy. The geothermal power station deals with thermal energy of the Earth. Places with large water areas or swift rivers are suitable for hydro-electric power stations. The last but not least possibility is to gain biomass energy. There are several other alternative energy sources but such as energy of sea waves are not reachable in the Czech conditions.

Each of the mentioned energy sources has its advantages and also disadvantages. In this paper we evaluate them from multicriteria point of view with respect to their investment effectiveness and environmental impact. It is a complex decision making problem with a small number of alternatives (five in our study) and many decision criteria.

## 2. Formulation of the problem

In this paper we evaluate five alternatives (renewable energy sources) with respect to five main groups of criteria and the total number of criteria is 16. Followed is a brief description of all alternatives and criteria.

***Alternatives***

The list of alternatives is given by energy sources that can be built in the Czech Republic. The used data are from the real projects and they were gained from [4]. The list of alternatives is as follows:

- *Wind power station* – a wind farm with four wind power stations, output 2300 kW, lifetime 20 years, costs CZK about 367 millions (1 EURO = 26 CZK approx.),
- *Photovoltaic power station* – 373 kW, lifetime 15 years, costs about CZK 30 millions,
- *Geothermal power station* – 300 kW, lifetime 30 years, costs about CZK 23 millions,
- *Hydro-electric power station* – 103 kW, lifetime 20 years, costs about CZK 10 millions,
- *Biomass energy* – 1500 kW, lifetime 20 years, costs about CZK 170 millions.

***Criteria***

As was mentioned earlier the criteria used for the analysis are divided into five main groups. Each group contains several sub-criteria.

1. *Technical criteria*
   - *T1:* annual utilization of installed energy output (maximize) – a ratio of real energy production per year and maximal theoretical energy production,
   - *T2:* expected dissipation of energy (minimize) – dissipation of energy given by unstable weather, device errors, repairs and services,
   - *T3:* expected lifetime of power station in years (maximize),
   - *T4:* investment costs of the project (minimize) – capital expenditures related with building, complexity of realization, time of realization and technical complexity (evaluated by points from 0 to 100).
2. *Economic criteria*
   - *F1:* the net present value – NPV (maximize),

- *F2:* internal rate of return – IRR (maximize),
- *F3:* time of recovery (minimize),
- *F4:* recovery of investment – ROI (maximize),
- *F5:* net profit (maximize).

3. *Social criteria*
   - *S1:* number of created work positions evaluated in scale from 0 to 100 (maximize),
   - *S2:* user's comfort (maximize) – costingness of services, quality and complexity, evaluated by points from 0 to 100.

| Criterion | Measurement unit | Type | Wind | Photovoltaic | Geothermal | Hydro | Biomass |
|---|---|---|---|---|---|---|---|
| T1 | % | max | 1.009 | 0.093 | 0.799 | 0.593 | 0.856 |
| T2 | % | min | 0.080 | 0.097 | 0.040 | 0.178 | 0.144 |
| T3 | point | max | 20 | 15 | 30 | 20 | 20 |
| T4 | point | min | 75 | 40 | 95 | 80 | 60 |
| F1 | CZK | max | 119.6 | 1.0 | 94.7 | 7.0 | 366.9 |
| F2 | % | max | 0.115 | 0.080 | 0.407 | 0.157 | 0.307 |
| F3 | year | min | 8.5 | 8.5 | 2.5 | 6.5 | 3.5 |
| F4 | % | max | 2.034 | 1.529 | 11.090 | 2.158 | 0.671 |
| F5 | CZK | max | 386.5 | 28.4 | 103.7 | 11.408 | 86.8 |
| S1 | point | max | 65 | 40 | 75 | 70 | 55 |
| S2 | point | max | 70 | 80 | 75 | 75 | 55 |
| E1 | Kg/kWh | max | 2.917 | 0.268 | 2.310 | 1.715 | 2.475 |
| E2 | point | min | 80 | 75 | 50 | 60 | 60 |
| E3 | point | min | 40 | 25 | 10 | 50 | 40 |
| G1 | point | max | 40 | 85 | 25 | 40 | 75 |
| G2 | point | max | 65 | 70 | 90 | 45 | 60 |

Table 1 – The data set

4. *Environmental criteria*
   - *E1:* decreasing of carbon dioxide (maximize) – evaluated by amount of saved fugitive emissions in comparison with pitcoal,
   - *E2:* scenery derogation (minimize) – evaluated by points from 0 to 100,
   - *E3:* other environmental impacts (minimize) – such as audible noise, dustiness, effluvium, appropriation of land, etc., also evaluated by points from 0 to 100.
5. *Strategic group includes two sub-criteria*
   - *G1:* accessibility of suitable areas (maximize) – accessibility and suitability of area, adequacy of natural environment,
   - *G2:* volume of sources diversification (maximize) – evaluation of increase in number of energy mix sources.

Complete data set for MCDM analysis of investments into five renewable energy sources in the Czech Republic is included in Table 1.

## 3. Methods for MCDM evaluation of alternatives

For multicriteria evaluation of alternatives several different methodological approaches can be used. One of the most suitable is the analytic hierarchy process (AHP) which allows structuring complex decision problems into several interconnected hierarchical levels. In our study we use a four-level hierarchy with the following levels: main objective of the evaluation, main groups of criteria, particular criteria and alternatives on the last level of the hierarchy (Figure 1). The results of the evaluation strongly depend on preferences of the decision maker, i.e. on the weights of the criteria. That is why the analysis of renewable energy sources can be done from several points of view, e.g. investor's point of view with the main emphasis on economic criteria, regional authorities' point of view with the emphasis on social and environmental criteria, etc.

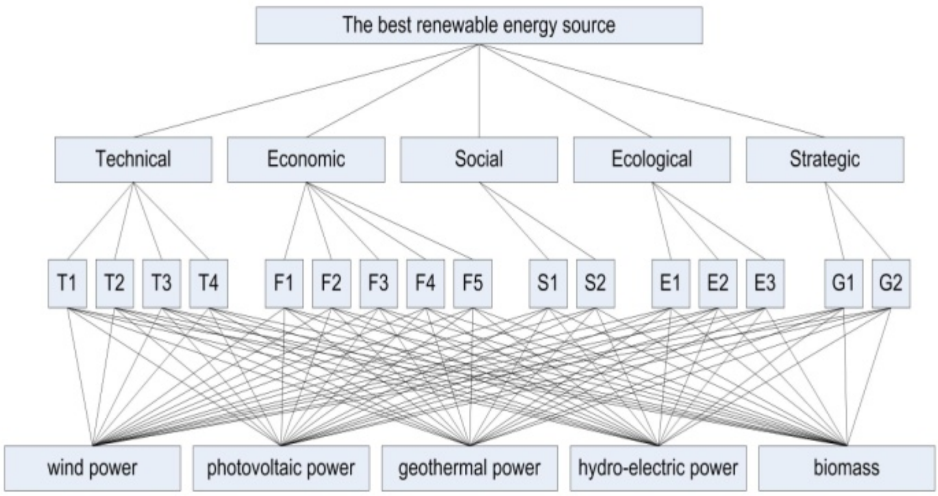


Figure 1. AHP model for renewable energy sources evaluation

The AHP is a primary tool for evaluation of energy sources in our study. The results given by this method will be compared by the results of other MCDM techniques in the next section of the paper. Except the AHP we use WSA (Weighted Sum Approach), TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), ELECTRE (ELimination Et Choix Traduisant la REalité) and PROMETHEE (Preference Ranking Organization METHod for Enrichment Evaluations) class methods. The AHP derives final priorities of alternatives by pairwise comparisons of the elements on particular levels of the hierarchy as described in detail e.g. in [6]. WSA sorts the alternatives by values of their utility functions which are supposed to be linear in this case. The WSA method requires information about the weights of the criteria. The basic concept of TOPSIS method consists in minimization of the distance from the ideal alternative and its maximization from the basal alternative. The decision maker must deliver information about the weights of criteria only. The ELECTRE I method [5] decides whether the alternative is effective or not. The decision maker must know the weights of the criteria, and the preference and dispreference thresholds. PROMETHEE [1] class methods use preference functions to express the intensity of preference for each pair of alternatives and for each criterion. The decision maker can choose among six types of intensity preference functions for each criterion. In this paper we use PROMETHEE II method which allows complete ranking of alternatives.

## 4. Computational experiments

As mentioned earlier the renewable energy sources can be evaluated from several points of view. Our main task is to evaluate them from the investor's point of view with higher emphasis on economic and technical criteria. In this section we will compare the investor's insight with the environmental insight with higher emphasis on environmental a social criteria. In the first step of application of the AHP the main groups of criteria are pairwise compared each other. In the same way, i.e. by pairwise comparisons, the weights of the groups of main criteria are divided further into particular criteria. The final results, i.e. the final weights of the criteria taking into account the investor's point of view, are presented in Table 2. The same approach was applied for estimation of weights of the criteria form the environmental point of view. The last two columns of Table 2 present the weights of the main groups of criteria and their splitting into particular criteria in this case.

| | Investor's point of view | | | Environmental view | |
|---|---|---|---|---|---|
| Main crit. | | Crit. | Weights | | Weights |
| Technical | 0.2701 | T1 | 0.1480 | 0.0853 | 0.0467 |
| | | T2 | 0.0573 | | 0.0181 |
| | | T3 | 0.0224 | | 0.0071 |
| | | T4 | 0.0424 | | 0.0134 |
| Economic | 0.5306 | F1 | 0.1523 | 0.0530 | 0.0152 |
| | | F2 | 0.1327 | | 0.0133 |
| | | F3 | 0.0578 | | 0.0058 |
| | | F4 | 0.0875 | | 0.0087 |
| | | F5 | 0.1003 | | 0.0100 |
| Social | 0.0806 | S1 | 0.0135 | 0.2175 | 0.0363 |
| | | S2 | 0.0671 | | 0.1812 |
| Environmental | 0.0831 | E1 | 0.0119 | 0.4697 | 0.0667 |
| | | E2 | 0.0356 | | 0.2015 |
| | | E3 | 0.0356 | | 0.2015 |
| Strategic | 0.0356 | G1 | 0.0305 | 0.1745 | 0.1495 |
| | | G2 | 0.0051 | | 0.0250 |

Table 2 – Weights of criteria derived by the AHP model

The two sets of weights are used for MCDM evaluation of five alternatives by means of several MCDM methods. The primary evaluation is done by standard AHP approach (pairwise comparisons of alternatives with respect to all criteria). The results of this evaluation are compared with the results given by other methods – WSA, TOPSIS, ELECTRE I and PROMETHEE II. The WSA and TOPSIS methods do not expect any knowledge of any additional information. ELECTRE I needs preference and dispreference thresholds – in our experiments these thresholds are set 0.8 and 1.0 respectively. PROMETHEE class methods expect the knowledge of one of six types of preference functions with their parameters for each criterion. We selected linear preference function with indifference area for each criterion. The results of evaluation for all five alternatives using five MCDM methods with two sets of weights reflecting investor's and environmental point of view are presented in Table 3. The utility values for the first three methods (AHP, WSA and TOPSIS) are normalized to unit sum for better comparability of the results. The best alternative(s) by each method is written in bold. All computations were done by means of specialized software tools for multiple criteria evaluation of alternatives developed at the Department of Econometrics, University of Economics Prague. The first of them is *Sanna* [2] which is built as MS Excel add-in application. The second one is *IZAR* [3] which is stand alone application.

| Methods | Wind | Photovoltaic | Geothermal | Hydro | Biomass |
|---|---|---|---|---|---|
| | | | *investor's point of view* | | |
| AHP | 0.206 | 0.124 | **0.330** | 0.094 | 0.246 |
| WSA | 0.212 | 0.098 | **0.313** | 0.122 | 0.255 |
| TOPSIS | 0.237 | 0.091 | 0.269 | 0.108 | **0.295** |
| ELECTRE | Ineff | Ineff | **Eff** | Ineff | **Eff** |
| PROMETH | 0.099 | -0.406 | **0.424** | -0.354 | 0.237 |
| | | | *environmental point of view* | | |
| AHP | 0.142 | 0.217 | **0.315** | 0.147 | 0.179 |
| WSA | 0.150 | 0.206 | **0.295** | 0.164 | 0.185 |
| TOPSIS | 0.109 | 0.203 | **0.324** | 0.157 | 0.207 |
| ELECTRE | Ineff | Ineff | **Eff** | Ineff | Ineff |
| PROMETH | -0.165 | -0.013 | **0.353** | -0.151 | 0.024 |

Table 3 – Results given by MCDM methods

## 5. Conclusions

Renewable energy sources integration may be the key element of new energy policy not only in the Czech Republic because it improves the stability and reliability of the energy system, minimizes environmental impact and significantly saves sources of limited and not ecological fossil fuels. The results presented in Table 3 show that the best alternative in our study is geothermal power station and it is interesting that this alternative reaches best evaluation by all methods and both sets of weights (except TOPSIS method taking into account investor's point of view). This result is not surprising. Geothermal power station reaches very good values by almost all criteria used in the evaluation (see Table 1). By higher reflecting investor's point of view the next two best alternatives are biomass and wind power stations and the two worse are the remaining ones, i.e. photovoltaic and hydroelectric power stations. It is quite surprising because the real situation in the Czech Republic is that especially photovoltaic power stations are very popular among investors. The reason consists probably in a special governmental support of this kind of energy in the last years which is not included in the available data set. Different results are given by taking into account the environmental set of weights. The best alternative is geothermal power station again but the ranking of remaining alternatives is quite different. Photovoltaic power station reaches the second place and it is followed by biomass energy. The worse two alternatives are wind and small hydroelectric power stations in this case.

**Acknowledgements**

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# THE COMPARISON OF THE CZECH HEALTH INSURANCE COMPANIES EFFICIENCY

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**Abstract:** The aim of the paper is comparison of the efficiency of the health insurance companies in the Czech Republic according to the selected criteria (from the year 2008) and also with the respect to the special services (above standard insurance) that the companies offer and pay. For the comparison the data envelopment analysis (DEA) models and multicriteria evaluation of alternatives were chosen.

**Keywords:** Health Insurance Companies, Comparison, Data Envelopment Analysis, Multicriteria Evaluation of Alternatives.

## 1. Introduction

Everyone who is permanently resident in the Czech Republic is obliged to participate in the health care insurance system, but the selection of the health insurance company is a free choice made by each person. The legal basis of the compulsory protection system in the Czech Republic consists of four laws, which cover the health insurance system (Act No. 48/1997 Collection of Law), health insurance (Act No. 492/1992 Collection of Laws), the General Insurance Company (Act No. 551/1991 Collection of Laws) and employee's insurance companies (Act No. 280/1992 Collection of Laws). These laws were passed by Parliament in 1991 and 1992 [15].

The public health insurance system in the Czech Republic is based on mutually binding relationships: **the insured – healthcare providers –**

**health insurers.** Czech healthcare system is inspired by the European tradition, founded on public services and financed by predominantly public means. It is provided predominately on the basis of obligatory public health insurance [23].

## 2. Health Insurance Companies

The largest of the public health insurers is the General Health Insurance Company of the Czech Republic (Všeobecná zdravotní pojišťovna ČR – hereinafter **VZP –** www.vzp.cz).

Aside from it there were nine other insurers in the year 2008 [12], but AGEL Health Insurance Company came into existence during the given year, so we compared these other eight:

- Health Insurance Company of the Ministry of Interior of the CR (Zdravotní pojišťovna ministerstva vnitra – ZPMVCR – www.zpmvcr.cz),
- Metal Aliance Health Insurance Company (Pojišťovna Metal Aliance www.zpma.cz),
- Czech National Health Insurance Company (Česká národní zdravotní pojišťovna – ČNZP – www.cnzp.cz)
- Business Health Insurance Company of Bank, Insurance and Building Employees (Oborová zdravotní pojišťovna zaměstnanců bank, pojišťoven a stavebnictví – OZP – www.ozp.cz).
- Metalurgical Health Insurance Company (Hutnická zdravotní pojišťovna – HZP – www.hzp.cz)
- Revírní Bratrská Pokladna – Health Insurance Company (RBP-ZP – www.rbp-zp.cz)
- Military Health Insurance Company of the Czech Republic (Vojenská zdravotní pojišťovna České republiky – VOZP – www.vozp.cz)

- Skoda Health Insurance Company (Zaměstnanecká pojišťovna Škoda – ZPŠ – www.zpskoda.cz)

## 3. Methodology

There exist several approaches for evaluation of insurance companies. One of them is the multiattribute evaluation of alternatives. As we presented in [20] a lot of methods can be used for selection of the best insurance company. There exist many methods for solving such problem, e.g. weighted sum approach – WSA, TOPSIS, ELECTRE, PROMETHEE, MAPPACC etc. [13], [14]. Unfortunately, the results can be generally sensitive to choice of weights.

The second approach is the using of multiple objective linear programming (MOLP) methods [19]. For solution well known methods, such as goal programming, methods of minimal distance from ideal solution, minimal component method [18], multicriterion simplex method [24] and also interactive methods as Steuer's method, Zionts-Wallenius' method etc. can be used [13], [14]. As these methods are parametric and so they are generally sensitive to additional information.

The third possible approach is the data envelopment analysis (DEA). DEA is a set of non-parametric techniques based on solving of linear programming problems for evaluation of efficiency of the set of homogenous units. The good insurance company consumes minimal amount of inputs and produces maximal amount of outputs. The basic idea of DEA models consists in estimation of an efficient frontier that defines production possibility set of the problem. The units lying on the frontier are considered as efficient and the remaining ones as inefficient. Their efficiency score is measured as a distance from the efficient frontier [17]. Data envelopment analysis (DEA), developed by Charnes et al. [11], has become one of the most widely used methods for evaluation of the relative efficiency of comparable units (DMU). Based on the set of available DMUs DEA estimates so-called efficient frontier, and projects all DMUs onto this frontier. If a DMU lies on the frontier, it is referred to as an efficient unit, otherwise inefficient. DEA also provides efficiency scores and reference units for inefficient DMUs. Reference units are hypothetical units on the efficient frontier, which can be regarded as target units for inefficient units. A virtual reference unit is traditionally found in DEA by projecting the inefficient DMU radially onto the efficient frontier. The advantage of DEA models is a fact that efficiency evaluation is based on the data available without taking into account the decision-maker's preferences. All efficient DMUs are considered equally "good" [17]. Note also that DEA models are linear problems those can be solved classically for example by simplex method and the alternative possibility for solution is to solve dual problem. DEA models can be oriented to inputs or outputs. In the case input oriented models we assume fixed level of outputs (CCR-I), the output oriented model assumes fixed level of inputs and maximize level of outputs with respect to given inputs (CCR-O) [11]. These models are used if we assume constant return to scale. In the case of variable return to scale we work with BBC (Banker, Charnes, Cooper) models. The review and detailed information about DEA models can be found in [1] and [21].

## 4. Data analysis

For the comparison of the health insurance companies we use data envelopment analysis. Afterwards we compare the results with the solution obtained from multicriteria evaluation of alternatives [20]. For both comparisons we use data from the year 2008 because only for this year it was possible to find all the necessary information. The selected criteria for each health insurance company are:

- Number of employees per a thousand insured
- Number of insured
- Average insurance (in thousand crowns) per an insured person
- Average expenses (in thousand crowns) per an insured person
- Number of contract clinics
- Number of offices per a thousand of insured
- Percent of expenses for preventive care

All the necessary data which were taken from the web pages of the companies [2], [3], [4], [5], [6], [7], [8], [9], [10] and from the Czech Statistical Office [12] are in the Table 1.

**Tab. 1** Data for the analysis

| Alternative \ Criterion | No. of employee / a thousand insured | No. of insured | Avg. insurance (thous. crowns)/ an insured person | Avg. expenses (thous. crowns) / an insured person | No. of contract clinics | No. of offices / a thousand of insured | Percent of expenses for preventive care |
|---|---|---|---|---|---|---|---|
| **HZP** | 0.667 | 362 615 | 18.276 | 16.098 | 8 193 | 0.113 | 0.051 |
| **ČNZP** | 0.689 | 307 713 | 18.586 | 16.346 | 20 462 | 0.208 | 0.004 |
| **VZP** | 0.744 | 6 429 707 | 21.697 | 20.209 | 34 160 | 0.029 | 0.004 |
| **ZPMA** | 0.581 | 385 516 | 16.272 | 14.593 | 7 872 | 0.106 | 0.031 |
| **OZP** | 0.523 | 672 992 | 17.835 | 16.948 | 22 706 | 0.025 | 0.009 |
| **RBP-ZP** | 0.541 | 366 160 | 15.962 | 13.292 | 5 568 | 0.101 | 0.010 |
| **VOZP** | 0.630 | 565 411 | 18.471 | 16.406 | 21 571 | 0.028 | 0.003 |
| **ZPŠ** | 0.647 | 131 332 | 18.872 | 16.556 | 2 960 | 0.038 | 0.006 |
| **ZPMVČR** | 0.569 | 1 100 551 | 17.801 | 16.075 | 22 018 | 0.077 | 0.022 |

In our analysis the number of employees per a thousand insured, the average insurance and the average expenses per one insured were taken into account on the input side. As outputs we assume the number of insured, the number of contract clinics, the number of offices and per cent of preventive care expenses.

The results are included in Table 2. We can see that all DEA models concluded HZP, ČNZP, VZP and ZPMVČR as efficient. By the super efficiency we can sort these insurance companies. For example in the case of CCR-I model the best instance company is VZP.

Remaining five companies are inefficient. From the table 2 we can see that OZP is almost efficient compared to ZPŠ. Note that VOZP and ZPŠ are dominated alternatives and so they cannot be efficient. Note also that VOZP is dominated but it is more efficient than non-dominated RBP-ZP. Further, for example ZPŠ is inefficient because it has too high inputs and very low outputs. In order to be efficient ZPŠ has to decrease all its inputs to 26% or increase all outputs almost four times (CCR models). Similarly we could interpret the other numbers.

If we assume variable return to scale (BCC models) the other three insurance companies are efficient. They are ZPMA, OZP and RBP-ZP. Both dominated alternatives are still inefficient.

**Tab. 2**: Results of data envelopment analysis

| | | ***HZP*** | ***ČNZP*** | ***VZP*** | ***ZPMA*** | ***OZP*** | ***RBP-ZP*** | ***VOZP*** | ***ZPŠ*** | ***ZPMVČR*** |
|---|---|---|---|---|---|---|---|---|---|---|
| CCR-I | theta | 1,000 | 1,000 | 1,000 | 0,921 | 0,999 | 0,747 | 0,794 | 0,256 | 1,000 |
| | super efficiency | 1,507 | 1,850 | 4,793 | | | | | | 1,193 |
| CCR-O | theta | 1,000 | 1,000 | 1,000 | 1,085 | 1,001 | 1,339 | 1,259 | 3,903 | 1,000 |
| | super efficiency | 0,664 | 0,540 | 0,209 | | | | | | 0,838 |
| BCC-I | theta | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 0,975 | 0,846 | 1,000 |
| | super efficiency | infeas. | infeas. | infeas. | 1,057 | 1,106 | 1,098 | | | 1,218 |
| BCC-O | theta | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,066 | 3,806 | 1,000 |
| | super efficiency | 0,602 | 0,515 | 0,171 | 0,517 | infeas. | infeas. | | | 0,770 |

By these results we can sort all analyzed insurance companies according to average place. The best and so the most efficient insurance company in the Czech Republic is VZP. The second place has ČNZP and the third one has HZP. On the other places are ZPMVČR, OZP, ZPMA, VOZP and RBP-ZP. ZPŠ is placed as the last.

As we analyzed the same data by multicriteria evaluation of alternatives [20] we can compare the results. You can see the order in Table 3. As VOZP and ZPŠ were dominated by ZPMVČR, it is clear that these companies must be worse than ZPMVČR – and in both comparisons they are in the back. But the ranking of the other companies is completely different when using

DEA models and multicriteria evaluation of alternatives (the order comes from the results of 5 methods – WSA, TOPSIS, ELECTRE, PROMETHEE, MAPPACC). It can be caused by the different techniques when DEA uses inputs and outputs but multicriteria evaluation alternatives uses all the criteria together and also it needs weights [16] – and this fact can cause the main differences. But we have found out that the order made by TOPSIS method is very close to the DEA results.

## 5. Conclusion

The dilemma which health insurance company to choose still belongs to the actual problems. The health insurance companies are engaged in so called general health insurance, it means necessary health services without direct payment in the range given by law. But their management can be different. Most methods in multicriteria evaluation of alternatives indicated that VZP (as the biggest health insurance company) is not so good as the others – but the results of the DEA models showed that in case of selected inputs and outputs it is still effective and good. As we know that HZP and ČNZP incorporated in one body (in 2009) into ČPZP (Czech Industrial Health Insurance Company), this can be a big competitor to VZP. This analysis shows big differences among the management of the health insurance companies. DEA analysis showed that the "bigger" companies can (VZP) but need not be effective (OZP) but if we add weights to the selected criteria some "small" companies (RBP-ZP) could be a good choice for the insured. So it is evident that it is complicated to choose the right health insurance company.

**Tab.3**: Comparison of the Results

| | Order according to DEA models | Order according to multicriteria evaluation of alternatives (avg. of 5 methods) | Order according to multicriteria evaluation of alternatives - TOPSIS |
|---|---|---|---|
| **HZP** | 3 | 5 | 6 |
| **ČNZP** | 2 | 4 | 2 |
| **VZP** | 1 | 7 | 1 |
| **ZPMA** | 6 | 1 | 5 |
| **OZP** | 5 | 6 | 7 |
| **RBP-ZP** | 8 | 3 | 4 |
| **VOZP** | 7 | 8 | 8 |
| **ZPŠ** | 9 | 9 | 9 |
| **ZPMVČR** | 4 | 2 | 3 |

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# NONLINEAR FUNCTIONALS IN STOCHASTIC PROGRAMMING; A NOTE ON STABILITY AND EMPIRICAL ESTIMATES

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**Abstract:** Economic processes are very often influenced simultaneously by a decision parameter (that can be chosen according to conditions) and a random factor. Since mostly it is necessary to determine the decision parameter without knowledge of a random element realization, a deterministic optimization problem has to be defined. This deterministic problem can usually depend on an “underlying" probability measure corresponding to the random element. The investigation of such types problems often belong to the stochastic programming field. The great attention has been focus on the problems in which objective functions depend “linearly" on the probability measure. This note is focus on the cases when the above mentioned assumption is not fulfilled; see e.g. Markowitz functionals or some risk measures. We try to cover static (one stage problems) as well as dynamic approaches (multistage stochastic programming case).

**Keywords:**Optimization problems with a random element, one stage stochastic programming problems, multistage stochastic programming problems, linear and nonlinear functionals, risk measures.

## 1. Introduction

Optimization problems depending on a probability measure correspond to many applications. They can be often investigated in the framework of the stochastic programming theory; in one–stage as well as in multistage settings. Objective functions are there mostly a linear “functional" of the “underlying" probability measure. However, it happens relatively often that this assumption is not fulfilled (see e.g. [6], [9]). In this note, we focus on this nonlinear case. First, we recall some corresponding one–stage problems, furthermore we try to generalize the definition and corresponding results to the multistage case.

## 2. One–Stage Stochastic Programming Problems

We start with a “classical" one–stage problem. To this end let $(\Omega, S, P)$ be a probability space; $\xi (:= \xi(\omega) = [\xi_1(\omega), \ldots, \xi_s(\omega)])$ an $s$–dimensional random vector defined on $(\Omega, S, P)$; $F (:= F(z), z \in R^s)$ the distribution function of $\xi$; $P_F, Z(:= Z_F)$ the probability measure and support corresponding to $F$. Let, moreover, $g_0 (:= g_0(x, z))$ be a real–valued (say continuous) function defined on $R^n \times R^s$; $X \subset R^n$ be a nonempty “deterministic" set. If the symbol $\mathrm{E}_F$ denotes the operator of mathematical expectation corresponding to $F$, then many economic applications (considering with respect only to one time point) can be introduced as the problem:

Find

$$\phi(F) = \inf\{\mathrm{E}_F g_0(x, \xi) \mid x \in X\}. \quad (1)$$

Evidently, the objective function in (1) depends linearly on the probability measure $P_F$. However, some applications correspond to optimization problems in which this assumption is not fulfilled. Let us consider the following very simple portfolio problem.:

Find

$$\max \sum_{k=1}^{n} \xi_k x_k \quad \text{s.t.} \quad \sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1,\ldots,n, \quad s = n, \tag{2}$$

where $x_k$ is a fraction of the unit wealth invested in the asset $k$, $\xi_k$ denotes the return of the asset $k \in \{1, 2,\ldots n\}$. If $\xi_k, k = 1,\ldots,n$ are known, then (2) is a linear programming problem. However, $\xi_k, k = 1,\ldots,n$ are mostly random variables with unknown realizations in a time decision. If we denote

$$\mu_k = \mathrm{E}_F \xi_k, \quad c_{k,j} = \mathrm{E}_F (\xi_k - \mu_k)(\xi_j - \mu_j), \quad k, j = 1,\ldots n, \tag{3}$$

then it is reasonable to set to the portfolio selection two–objective optimization problem:

Find

$$\max \sum_{k=1}^{n} \mu_k x_k, \quad \min \sum_{k=1}^{n} \sum_{j=1}^{n} x_k c_{k,j} x_j \quad \text{s. t.} \quad \sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1,\ldots,n, \tag{4}$$

where $\sum_{k=1}^{n} \sum_{j=1}^{n} x_k c_{k,j} x_j$ can be considered as a risk measure.

Evidently, there exists only rarely a possibility to find an optimal solution simultaneously with respect to the both criteria. Markowitz suggested (see e.g. [2]) to replace the problem (4) by one–criterion optimization problem of the form:

Find

$$\phi^M(F) = \max\{\sum_{k=1}^{n} \mu_k x_k - K\sum_{k=1}^{n}\sum_{j=1}^{n} x_k c_{k,j} x_j\} \quad \text{s. t.} \quad \sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1,\dots,n, \tag{5}$$

where $K \ge 0$ is a constant.

Konno and Yamazaki introduced in [7] another risk measure $w(x)$ by

$$w(x) = \mathrm{E}_F \,|\sum_{k=1}^{n} \xi_k x_k - \mathrm{E}_F[\sum_{k=1}^{n} \xi_k x_k]\,|\,. \tag{6}$$

Some other suitable risk measures can be found e.g. in [9].

Evidently, $w(x)$ is a Lipschitz function of $\mathrm{E}_F[\sum_{k=1}^{n} \xi_k x_k]$ and, consequently, the problem

Find

$$\max\{\lambda \sum_{k=1}^{n} \mu_k x_k - (1-\lambda)\mathrm{E}_F \,|\sum_{k=1}^{n} \xi_k x_k - \mathrm{E}_F[\sum_{k=1}^{n} \xi_k x_k]\,|, \quad \lambda \in [0,1] \tag{7}$$

can be covered by the more general problem:

Find

$$\phi(F) := \overline{\phi}(F) = \inf\{\mathrm{E}_F g_0^1(x, \xi, \mathrm{E}_F h(x,\xi)) \,|\, x \in X\}, \tag{8}$$

where $h(x,z) = (h_1(x,z),\dots,h_{m_1}(x,z))$ is $m_1$–dimensional vector function defined on $R^n \times R^s$, $g_0^1(x,z,y)$ is a real–valued (say uniformly continuous) function defined on $R^n \times R^s \times R^{m_1}$.

## 3. Multistage Stochastic Programming Problems

Many real–life problems with a random factor, those are developing over time, can be treated by multistage stochastic techniques. To this end let random factors $\xi^k$ and decisions $x^k$ $k = 0,1,\dots$ follow the scheme:

$$x^0 \longrightarrow \xi^0 \longrightarrow x^1 (:= x^1(\xi^0, x^0)) \longrightarrow \xi^1 \longrightarrow x^2 (:= x^2(\bar{\xi}^1, \bar{x}^1)) \longrightarrow \xi^2 \longrightarrow \ldots \ \ldots \longrightarrow$$
$$x^{M-1} (:= X^{M-1}(\bar{\xi}^{M-2}, \bar{x}^{M-2})) \longrightarrow \xi^{M-1} \longrightarrow x^M (:= x^M(\bar{\xi}^{M-1}, \bar{x}^{M-1})) \longrightarrow \xi^M \ldots, \tag{9}$$

where $\bar{x}^k = [x^0, x^1, \ldots, x^k], \quad \bar{\xi}^k = [\xi^0, \xi^1, \ldots, \xi^k], \quad k = 0, 1, \ldots$

Evidently, it follows from the relation (9) that for every $k = 0, 1, \ldots$ the decision $x^k$ can depend on $x^0, \ldots, x^{k-1}$ and $\xi^0, \ldots \xi^{k-1}$, however it can not depend on $x^{k+1}, \ldots$ and $\xi^k, \ldots$ We say that the decision has to be nonanticipative.

Considering the above mentioned situation with respect to a discrete time interval $[0, M]$ and supposing that the decision parameter can be determined with respect to the average of a corresponding objective function, we can set usually to the relation (9) a “classical" multistage ($M+1$–stage) stochastic programming problem (for more details see e.g. [1] or [10]):

Find

$$\phi_F(M) = \inf \{\mathrm{E}_{F^{\xi^0}} g_F^0(x^0, \xi^0) \mid x^0 \in K^0\}, \tag{10}$$

where the function $g_F^0(x^0, z^0)$ is defined recursively

$$g_F^k(\bar{x}^k, \bar{z}^k) = \inf\{\mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}} g_F^{k+1}(\bar{x}^{k+1}, \bar{\xi}^{k+1}) \mid x^{k+1} \in K_F^{k+1}(\bar{x}^k, \bar{z}^k)\},$$
$$k = 0, 1, \ldots, M-1, \tag{11}$$
$$g_F^M(\bar{x}^M, \bar{z}^M) := g_0^M(\bar{x}^M, \bar{z}^M), \quad K_0 := X^0.$$

$\xi^j := \xi^j(\omega), \quad j = 0, 1, \ldots, M$ denotes an $s$–dimensional random vector defined on a probability space $(\Omega, S, P)$; $F^{\xi^j}(z^j), \ z^j \in R^s, j = 0, 1 \ldots, M$ the

distribution function of the $\xi^j$ and $F^{\xi^k|\bar{\xi}^{k-1}}(z^k \mid \bar{z}^{k-1})$, $z^k \in R^s, \bar{z}^{k-1} \in R^{(k-1)s}, k = 1,\ldots, M$ the conditional distribution function ($\xi^k$ conditioned by $\bar{\xi}^{k-1}$); $P_{F^{\xi^j}}, P_{F^{\xi^{k+1}|\bar{\xi}^k}}$, $j = 0,1,\ldots,M, k = 0,1,\ldots,M-1$ the corresponding probability measures; $Z^j := Z_{F^{\xi^j}} \subset R^s, j = 0,1,\ldots,M$ the support of the probability measure $P_{F^{\xi^j}}$. Furthermore, the symbol $g_0^M(\bar{x}^M, \bar{z}^M)$ denotes a uniformly continuous function defined on $R^{n(M+1)} \times R^{s(M+1)}$; $X^0 \subset R^n$ is a nonempty compact set; the symbol $K_F^{k+1}(\bar{x}^k, \bar{z}^k) := K^{k+1}_{F^{\xi^{k+1}|\bar{\xi}^k}}(\bar{x}^k, \bar{z}^k)$, $k = 0,1,\ldots,M-1$ denotes a multifunction mapping $R^{n(k+1)} \times R^{(k+1)}$ into the space of subsets of $R^n$.
$\bar{\xi}^k (:= \bar{\xi}^k(\omega)) = [\xi^0,\ldots,\xi^k]; \bar{z}^k = [z^0,\ldots,z^k], z^j \in R^s$; $\bar{x}^k = [x^0,\ldots,x^k]$, $x^j \in R^n$; $j = 0,1,\ldots,k$, $k = 0,1,\ldots,M$. Symbols $\mathrm{E}_{F^{\xi^0}}$, $\mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}}, k = 0,1,\ldots,M-1$ denote the operators of mathematical expectation corresponding to $F^{\xi^0}, F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}, k = 0,\ldots,M-1,$ $\quad F = \{F^{\xi^0(z^0, F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1})}), k = 1,\ldots,M\}.$

The problem (10) is a "classical" one–stage stochastic programming problem depending on the probability measure $P_{\xi^0}$, the problems (11) are for $k = 0,1,\ldots,M$ parametric one–stage stochastic programming problems depending on the conditional probability measures $P_{F^{\xi^{k+1}|\bar{\xi}^k}}$. Simultaneously, the objective functions depend "linearly" on the above mentioned measures. However, this assumption is not fulfilled every time; see the former section for one–stage case. Now we try to generalize one–stage case to the multistage approach. To this end we assume:

i.1 There exist $m_1$ –dimensional vector functions $\bar{h}^j(x^j, z^j) = (h_1^j(x^j, z^j), \ldots, h_{m_1}^j(x^j, z^j))$ defined on $R^n \times R^s$ and real–valued (say uniformly continuous) functions $\bar{g}_0^j(x^j, z^j, y^j)$ defined on $R^n \times R^s \times R^{m_1}$, $j = 0, 1, \ldots, M$ such that

$$g_0^M(\bar{x}^M, \bar{z}^M) := \sum_{j=0}^{M} \bar{g}_0^j(x^j, z^j, \mathrm{E}_{F^{\xi^j|\bar{\xi}^{j-1}=\bar{z}^{j-1}}} \bar{h}^j(x^j, \xi^j)). \tag{12}$$

The multistage problem (10), (11) then can be (according to (9)) replaced by the following problem with nonlinear objective functions:

Find

$$\bar{\phi}_F(M) = \inf\{\mathrm{E}_{F^{\xi^0}}[\bar{g}_0^0(x^0, \xi^0, \mathrm{E}_{F^{\xi^0}} \bar{h}^0(x^j, \xi^0)) + \bar{g}_F^0(x^0, \xi^0)] \,|\, x^0 \in K^0\}, \tag{13}$$

where the function $g_F^0(x^0, z^0)$ is defined recursively

$$\begin{aligned}
\bar{g}_F^k(\bar{x}^k, \bar{z}^k) &= \inf\{\mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}[\bar{g}_0^{k+1}(x^{k+1}, \xi^{k+1}, \mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}} \bar{h}^{k+1}(x^{k+1}, \xi^{k+1})) + \\
&\qquad \bar{g}_F^{k+1}(x^{k+1}, \xi^{k+1})] \,|\, x^{k+1} \in K_F^{k+1}(\bar{x}^k, \bar{z}^k)\}, \\
k &= 0, 1, \ldots, M-2, \\
\bar{g}_F^{M-1}(\bar{x}^{M-1}, \bar{z}^{M-1}) &:= \inf\{\mathrm{E}_{F^{\xi^M|\bar{\xi}^{M-1}}} \bar{g}_0^M(x^M, \xi^M, \mathrm{E}_{F^{\xi^M|\bar{\xi}^{M-1}=\bar{z}^{M-1}}}, \bar{h}^M(x^M, \xi^M)) \,| \\
&\qquad x^M \in K_F^M(\bar{x}^{M-1}, \bar{z}^{M-1})\}, \quad K^0 := X^0, \\
\bar{g}_F^M(\bar{x}^M, \bar{z}^M) &:= \bar{g}_0^M(x^M, z^M, \mathrm{E}_{F^{\xi^M|\bar{\xi}^{M-1}=\bar{z}^{M-1}}} \bar{h}^M(x^M, \xi^M)).
\end{aligned} \tag{14}$$

## 4. Problem Analysis

Of course the investigation of the problems (10), (11) or (13), (14) is very complicated. The stability (w.r.t. probability measure space) and empirical estimates (of the problem (10), (11)) have been investigated e.g. in [3], [5]. To investigate the problems (13), (14) we recall corresponding results for one–stage case. To this end let $P(R^s)$ denote the set of Borel probability measures on $R^s, s \geq 1$ and let

$$M_1(R^s) = \{P \in P(R^s) : \int_{R^s} \|z\|_s^1 P(dz) < \infty\}, \quad \|\cdot\|_s^1 \quad \text{denotes} \quad L_1 \quad \text{norm in} \quad R^s.$$

We introduce the assertion proven in [6], based on the approach employed in [4].

**Proposition 1.** [6] Let $X$ be a compact set, $G$ be an arbitrary $s-$ dimensional distribution function. Let, moreover, $P_F, P_G \in M_1(R^s)$. If

1. $g_0^1(x,z,y)$ is for $x \in X, z \in R^s$ a Lipschitz function of $y \in Y$ with a Lipschitz constant $L^y$; $Y = \{y \in R^{m_1} : y = h(x,z) \quad \text{for some} \quad x \in X, z \in R^s\}$,

2. for every $x \in X, y \in Y$ there exist finite mathematical expectations $\mathrm{E}_F g_0^1(x,\xi,\mathrm{E}_F h(x,\xi))$, $\mathrm{E}_F g_0^1(x,\xi,\mathrm{E}_G h(x,\xi))$, $\mathrm{E}_G g_0^1(x,\xi,\mathrm{G}_F h(x,\xi))$,

3. $h_i(x,z), i = 1,\ldots,m_1$ are for every $x \in X$ Lipschitz functions of $z$ with the Lipschitz constants $L_h^i$ (corresponding to $L_1$ norm),

4. $g_0^1(x,z,y)$ is for every $x \in X, y \in R^{m_1}$ a Lipschitz function of $z \in R^s$ with the Lipschitz constant $L^z$ (corresponding to $L_1$ norm),

then there exist $\hat{C} > 0$ such that

$$|\overline{\phi}(F) - \overline{\phi}(G)| \leq \hat{C} \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)|\, dz_i. \tag{15}$$

Evidently, the assertion of Proposition 1 can be employed for the investigation of empirical estimates of the problem (8) (for more details see [6]). There has been proven that convergence rates of the problems (1), (8) are (under the corresponding assumptions) the same. They can depend on the tails of one–dimensional marginals distribution functions.

To investigate the problem (13), (14) we introduce a system of the next assumptions:

i.2 There exists a random vector $\varepsilon^k := \varepsilon^k(\omega)), k = \ldots, -1, 0, 1, \ldots$ defined on $(\Omega, S, P)$

such that:

$\xi^0, \varepsilon^k$ (defined on $(\Omega, S, P), k = 1, 2, \ldots$ are stochastically independent,

$\varepsilon^k, k = 0, 1\ldots$ are identically distributed. (We denote the distribution function corresponding to $\varepsilon^1$ by the symbol $F^\varepsilon$),

i..3 there exists a Lipschitz vector ($s$–dimensional) function $H = [H_1, ..., H_s]$ defined on $R^s$ such that (for sequence of $s$–dimensional random vectors $\{\xi^k\}_{k=-\infty}^{\infty}$ one of the following conditions is valid

- $\xi^k = \varepsilon^k H(\xi^{k-1}), k = \ldots -1, 0, 1, \ldots, \quad \varepsilon^k = [\varepsilon_1^k, ..., \varepsilon_s^k], \quad \xi^k = [\xi_1^k, ..., \xi_s^k]$
- $\xi^k$ follows random sequence such that

$$\xi^k = \varepsilon^k + H(\xi^{k-1}), k = \ldots -1, 0, 1, \ldots,$$

i.4 the multifunction $K_F^{k+1}(\bar{x}^k, \bar{z}^k), k = 0, 1, \ldots, M-1$ do not depend on the system $F$.

A similar system of the assumptions have been already employed in [5], [8]. Employing the proofs technique of the paper [6] we can (under some additional assumptions) obtained (for problems (13), (14)) very similar results to them for one–stage case. Evidently, to this end it is necessary to find out assumptions under which the functions $\bar{g}_F^k(\bar{x}^k, \bar{z}^k), k = 0, \ldots, M$ are uniformly continuous and Lipschitz functions of $z^k$ with the Lipschits constant not depending on $\bar{x}^k, \bar{z}^{k-1}$. Furthermore the constraint sets $K_F^{k+1}(\bar{x}^k, \bar{z}^k)$, $k = 0, 1, \ldots, M-1$ have to be compact. To this end the approach of the papers [3], [5] can be employed. However, more detailed investigation is over the possibility of this note.

## 5. Conclusions

In the note we have tried to introduce some types of optimization problems in which objective functions are not linear “functionals" of the “underlying" probability measures. Furthermore, we tried to give a brief sketch of their stability and empirical estimates investigation. According to this, it is possible to see that the results in the case of linear dependence and some nonlinear case are the same for corresponding one–stage problems. Moreover, if the random element follows autoregressive random sequences (in the multistage case) we can obtain also very similar results for the multistage case.

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# LONG-RUN STRUCTURAL MACROECONOMETRIC MODELS OF THE SLOVAK AND CZECH ECONOMIES

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**Abstract:** The paper applies the modelling strategy developed in Garratt, Lee, Pesaran and Shin (2006) to estimate long-run structure macroeconometric models. The strategy provides a practical approach to incorporating theoretic long-run relationships in a structural vector error correction model. We apply this modelling approach to the similar transformed economies – Slovak and Czech. The model originated by Garratt et al for the UK is modified therefore. Our results confirm the similarity of both economies with expected differences caused by a long time conjoint economic progress.
**Keywords:** long-run structural model, purchasing power parity, interest rate parity, output relationship, money market, Fisher inflation parity, Harrod-Balassa-Samuelson effect.

## 1. Introduction

The aim of our paper is to modify and empirically verify the modelling strategy developed by Garratt, Lee, Pesaran and Shin (2006). We have already published our first version of the long-run structure macroeconometric models for the Slovak and Czech economies (see Hančlová, Lukáčik and Szomolányi, 2010), so we try to improve our models and to approximate them to a reality of the transformed economies.

These published small open economy models were based on the production technology and the output determination, the arbitrage conditions, the long-run solvency requirements and the accounting identities and the stock-flow relations. This leads to 5 long-run equations. Now, we use the same logic as the first time with the same number of co-integrating relations.

The domestic variables were the real money stock, the real gross domestic product, the nominal interest rate, the ratio of the domestic price level and the rate of inflation. Further endogenous variables were the nominal exchange rate and the ratio of the foreign price level, the foreign real GDP and the foreign interest rate. Now, we change this structure. Following Assenmacher-Wesche and Pesaran (2008), we assume that the forcing (weakly exogenous) variables are the foreign real output and the foreign interest rate.

## 2. Economic Theory of the Models

The long run aggregate output is determined according to the production function:

$$Y_t = A_t L_t F\left(\frac{K_t}{A_t L_t}, 1\right), \qquad (1)$$

where $Y_t$ is real gross domestic product in Euros, $A_t$ stands for an index of labour-augmenting technological progress. Input factors of the production function are labour $L_t$ and capital stock $K_t$; $K_t/A_t/L_t$ is the capital stock per effective labour unit; $f(k_t) = \mathrm{F}(K_t/A_t/L_t,1)$ is a well behaved function in the sense that it satisfies the Inada conditions. See, for example, Barro and Sala-i-Martin (1998, p. 16).

Let us focus on the input factors – labour and capital – and on the technological progress. Likewise the UK economy, the Slovak and Czech economies are open such that we can adapt assumption of Garratt et al that $A_t$ is determined by the level of technological progress in the rest of the world; namely:

$$A_t = \gamma A_t^* e^{u_{A,t}}, \qquad (2)$$

where $A_t^*$ represents the level of foreign (EU) technological progress, $\gamma$ captures productivity differentials based on fixed, initial technological endowments and $u_{A,t}$ represents stationary, mean zero disturbances capturing the effects of information lags or (transitory) legal impediments to technology flows across different countries, for example.

The long-run labour input depends on the population:

$$L_t = \lambda_t N_t, \qquad (3)$$

where $N_t$ is the population size. The rate $\lambda_t$ is the labour rate that expresses the relation between the population size and the labour input.

Both Slovak and Czech economies have transformed and reformed and has integrated to the European structures. We assume that the difference between the transition (Slovak or Czech) and foreign (European) steady state capital stocks per effective labour unit is determined by the risk premium of the economy, measured as the difference between domestic and foreign interest rates. Moreover the lower is the risk premium more capital flows from abroad. The risk premium captures a performance of the policy authorities that enforced to transform and integrate the economy. The increase of the steady state capital per labour unit (caused by the decrease of the risk rate) could endow other units of the labour. The increase of the steady state capital per effective unit towards European level causes the convergence of the economy in line with the growth theory (see Barro and

Sala-i-Martin). Another determinant of the capital and the labour rate is stochastic:

$$\lambda_t f\left(k_t\right) = \lambda_t^* f\left(k_t^*\right) \kappa_0 e^{\kappa_r\left(r_t - r_t^*\right)} e^{u_{\kappa,t}}, \qquad (4)$$

where $\lambda_t^* f(k_t^*)$ represents foreign factors inputting the production, $\kappa_0$ captures the initial input factor steady state differences. The log of the by-index-expressed nominal interest rate is $r_t$, i.e. $r_t = \ln(1 + R_t)$, where $R_t$ is nominal interest rate. The term $u_{\kappa,t}$ represents a stationary, mean-zero process capturing cyclical fluctuations of the economy around its steady-state value. Assuming that per capita output in the rest of the world is also determined according to a neoclassical growth model, and using a similar line of reasoning as Garratt et al we have:

$$y_t - y_t^* = \ln \gamma + \ln \kappa_0 + \kappa_i \left(r_t - r_t^*\right) + u_{A,t} + u_{\kappa,t}, \quad (5)$$

where $y_t = \ln(Y_t/N_t)$ and $y_t^* = \ln(Y_t^*/N_t^*)$.

The other long-run relations are from Garratt et al except the Purchasing Power Parity. We capture the "Harrod-Balassa-Samuelson effect in which the price of a basket of traded and non-traded goods rises more rapidly in countries with relatively rapid productivity growth in the traded goods sector" (Garratt, Lee, Pesaran and Shin, p. 71)[1] by assuming that Purchasing Power Parity depends on $y_t - y_t^*$. We can the long-run relations write in the form:

$$p_t - p_t^* - e_t = b_{10} + \beta_{14}\left(y_t - y_t^*\right) + \xi_{1,t+1}, \qquad (6)$$

$$r_t - r_t^* = b_{20} + b_{21}t + \xi_{2,t+1}, \qquad (7)$$

$$y_t - y_t^* = b_{30} + \beta_{32}\left(r_t - r_t^*\right) + \xi_{3,t+1}, \qquad (8)$$

$$h_t - y_t = b_{40} + b_{41}t + \beta_{42}r_t + \beta_{44}y_t + \xi_{4,t+1}, \qquad (9)$$

[1] Authors refer Obstfeld and Rogoff (1996, Chapter 4) and Rogoff (1996) for further discussion of this effect and alternative modifications to PPP.

$$r_t - \Delta\tilde{p}_t = b_{50} + \xi_{5,t+1}, \qquad (10)$$

In the system of the equations (6)-(10) we denote the foreign variables by asterisks. By the symbol $p_t$ we denote the log of the general price level; the log of the exchange rate defined as the domestic price of a unit of foreign currency is denoted by $e_t$. The log of per capita money stock is $h_t$, we denote time by $t$ and, finally, the long-run stochastic terms are $\xi_{1,t+1}$-$\xi_{5,t+1}$.

The equation (6) is the log-linear version of the (relative) purchasing power parity (PPP), the interest rate parity relation (IRP) is represented by the equation (7), the output gap by the equation(8), the money demand relation by the equation (9) and, finally, the equation (10) represents the Fisher inflation parity (FIP).

## 3. Estimation and Testing of the Models

Our model contains eight variables $\mathbf{z_t} = \begin{pmatrix} e_t & r_t & \Delta\tilde{p}_t & y_t & p_t - p_t^* & h_t - y_t & r_t^* & y_t^* \end{pmatrix}^T$ and the foreign output and the interest rate are considered to be long-run forcing variables (weakly exogenous). The outcome of this decision is straightforward; the long-run forcing variables appear in the co-integrating relations but without having to specify the equations for them. Of course, we also expect the effect of these variables in the short-run, therefore the first differences of foreign output and foreign interest rate together with the first differences of oil prices (Garratt et al primarily consider variable) complete the short-run determinants.

The sample data are quarterly but indeed relative short, that’s why we use just four lags as an upper frontier of VAR lag length. Hannan-Quinn information criterion confirms one lag length of VAR model for Slovak and two lags length of VAR model for Czech economy.

The lambda trace statistics reject the null hypotheses that the number of co-integrating relations is equal 0, 1, 2, 3 and 4 at the 5 % significance level, but cannot reject the null hypothesis that the number of co-integrating relations is equal 5 for both economies. We use VEC model with unrestricted intercepts and restricted trend coefficients

With five co-integrating vectors we could impose 25 restrictions to fully identify model. The co-integrating vectors obtained by exact identification are not presented here, since they don't have an economic interpretation. Having fully identified the long-run relations, we then tested the over-identifying restrictions predicted by the theory.

The co-integrating matrix $\boldsymbol{\beta}^T$ with 15 over-identifying restrictions (11 + 4 restrictions on trend) for the Czech economy takes the following form:

$$\boldsymbol{\beta}^T = \begin{pmatrix} 1 & 0 & 0 & \beta_{14} & -1 & 0 & 0 & -\beta_{14} \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \beta_{32} & 0 & -1 & 0 & 0 & -\beta_{32} & 1 \\ 0 & \beta_{42} & 0 & \beta_{44} & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (11)$$

The model of Slovak economy takes one more restriction $\beta_{44} = 0$, than the co-integrating matrix $\boldsymbol{\beta}^T$ has 16 over-identifying restrictions (12 + 4 restrictions on trend). The estimates of the error corrections coefficients show that every long-run relation makes a contribution in at least one equation and that the error correction model with these long-run relations is reasonable. The results also show that the exogenous variables are responsible regressors.

## 4. Estimation and Testing of the Long-run Relations

The estimated long-run relations incorporated all the restriction suggested by theory with trend restrictions reflected the Slovak particularities and accepted a unitary income elasticity of money demand (restriction $\beta_{44} = 0$) take the form:

$$
\begin{aligned}
pps_t - e_t &= -1.78892 + 1.2134\left(y_t - y_t^*\right) + \xi_{1,t+1} \\
r_t - r_t^* &= 0.15404 - 0.0023147t + \xi_{2,t+1} \\
y_t - y_t^* &= -0.97534 - 4.7198\left(r_t - r_t^*\right) + \xi_{3,t+1} \qquad (12)-(16) \\
h_t - y_t &= 1.33808 - 8.1808r_t + \xi_{4,t+1} \\
r_t - \Delta\tilde{p}_t &= 0.116937 + \xi_{5,t+1}
\end{aligned}
$$

The long-run equations of model of Czech economy take the form:

$$
\begin{aligned}
pps_t - e_t &= -1.62588 + 1.4819\left(y_t - y_t^*\right) + \xi_{1,t+1} \\
r_t - r_t^* &= 0.037347 - 0.00395884t + \xi_{2,t+1} \\
y_t - y_t^* &= -1.21872 - 1.255125\left(r_t - r_t^*\right) + \xi_{3,t+1} \qquad (17)-(21) \\
h_t - y_t &= -4.24611 - 0.958675r_t + 0.69358y_t + \xi_{4,t+1} \\
r_t - \Delta\tilde{p}_t &= 0.036634 + \xi_{5,t+1}
\end{aligned}
$$

The convergence of the Slovak and Czech economies on the European economy becomes evident by the long-run average quarterly decrease in the risk premium by about $-b_{21}$, (i.e. 0.231% in Slovak economy and 0.396% in Czech economy) which positively affects the output gap. A one-percent average quarterly decrease of the risk premium affects a long-run increase of the output gap by about $-\beta_{32}$ (4.72% Slovak and 1.255% Czech), which positively affects the purchasing power parity. A one-percent average quarterly increase of the output gap affects a long-run increase of the PPS by about $\beta_{14}$ (1.213% Slovak and 1.482 Czech). The estimation of the average long-run Slovak real interest rate; $\exp(b_{50}) - 1 = 12.4$ %; is sort of problematic. The Czech average long-run Slovak real interest rate is about 3.731%. Model better matches the Slovak data, if the $\beta_{44}$ coefficient vanishes. We assume that the long-run elasticity of the influence of the output on the money demand is 1, following the quantitative theory of the money. The long-run speculative money demand elasticity is $\beta_{42} = -1.898$. The Czech long-run speculative money demand elasticity is $\beta_{42} = -0.959$ and transaction money demand elasticity is $\beta_{44} = 0.694$. Model better matches the both Slovak and Czech data, either if we assume no trend in money demand equations. Unlike Garratt et al, we measure the money stock using the M2 in Czech and M1 in Slovak case. The M2 and M1 aggregates could be independent on the changing nature of financial intermediation, and the increasing use of credit cards in settlement of transactions (unlike the high-powered M0 money). Therefore, we do not need trend in the money demand equation.

The over-identifying restrictions are tested by the log-likelihood ratio statistics which takes the value 40.0718 for the Slovak model and 93.6463 for the Czech model. The test statistics is asymptotically distributed as a $\chi 2$ variate with 16 degrees of freedom (15 degrees of freedom for Czech model). We don't make the conclusion directly, because the works by Haug (1996) and Abadir et al (1999) shown that the asymptotic critical values may not be valid for vector autoregressive models with a relatively large number of variables, unless samples are sufficiently large, what is just our case.

That's why we decided to implement the significance test of the log-likelihood ratio statistics using critical values which are computed by non-parametric bootstrap techniques with 5000 replications. For each replication, an artificial data set of endogenous variables is created by resampling with replacement of residuals computed from initial estimation. The test is carried out on each of the replicated data sets and the distribution of the statistics is derived across all replications. This shows that the relevant critical values for the test statistics are 90.8463 at the 5 % significance level and 85.145 at the 10 % significance level for the Slovak model and 116.2505 at the 5 % significance level and 109.354 at the 10 % significance level for the Czech model. We cannot therefore reject the over-identifying restrictions implied by the theory for both models.

## 5. Conclusion

By comparing of the estimated parameters of both Slovak and Czech models we can express interesting conclusions about both economic structures. The Slovak economic growth has been one of the greatest in the Europe. Slovak politics (who are actually in government) has been often using this fact in the reasoning of their performance. However, we consider the decrease of the risk premium as the size of the economic policy performance and not the economic growth. The risk premium effect against the output gap is higher in Slovakia than in Czech Republic. This is in line with the growth theory (see Barro and Sala-i-Martin). The Slovak transition state is lower and therefore the same growth in the steady state (decrease of the risk premium) causes higher growth in Slovak economy than in Czech one. However, the risk premium is decreasing faster – and so the steady state per capita output is growing higher – in Czech economy than in Slovak one. The average long-run quarterly growth of the output gap ($b_{21}$ x $\beta_{32}$) is higher in the Slovak economy (1.092%) than in Czech one (0.497%) which is in line with the observations of the Slovak politics. Likewise, the average long-run quarterly growth of the PPS is higher in Slovak economy (1.326%) than in Czech one (0.736%).

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# ROUTE ASSIGNMENT BASED ON K-SHORTEST PATHS PROBLEM

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**Abstract:** An OD matrix is a matrix which contains at the place $(i, j)$ the number of passengers requiring traveling from origin $v_i$ into destination $v_j$. In a real transportation network there are several routes how to get from $v_i$ to $v_j$. However OD matrix does not describe how the passengers are distributed along these routes. In this paper we present a method to assess the loads of transport network links making use of $k$-shortest path algorithm.

**Keywords:** OD matrix, traffic assignment, route assignment, shortest path.

## 1. Introduction

Full knowledge of passenger travel is an essential precondition of a line system design for a municipal passenger transport. The numbers of passengers travelling from origin $v_i$ to destination $v_j$ are contained in OD matrix. In practice, the OD matrix is estimated in either of the two following ways:

- complete traffic count
- calculation using another data (data from history, partial traffic count etc.)

Contemporary electronic tariff systems provide a lot of exact data about travelling of passengers. However, those data do not contain passenger destinations in the case of municipal transport.

Papers [4], [5] present several possibilities how to estimate destination bus stops in municipal passenger transport, i.e. how to asses elements of desired OD matrix.

Several optimization methods use flows along segments of transportation network instead of elements of OD matrix – the result of so called traffic assignment which can be estimated from correspondent OD matrix. Method PRIVOL [2] is an example of such procedures. The result of method PRIVOL is an assignment of available vehicles to particular lines with objective to maximize capacity reserve on segments of transportation network.

This paper is devoted to two possibilities of traffic assignment making use of OD matrix. We will use several point to point shortest path algorithms. The results will be demonstrated using data from electronic tariff system for March 2006 of municipal passenger bus transport Martin – Vrútky.

### 1.1. Fundamental Algorithm

The fundamental algorithm for traffic assignment was formulated by Černý and Kluvánek in [2].

This algorithm needs the following input data:

- Model of transportation network – a directed graph $G = (V, A, c)$, where $V$ is a set of nodes of transportation network, $A$ is a set of directed arcs – segments of transportation network and $c(a) \geq 0$ is the length of the segment . The length of a segment can express its mileage, duration of a travel along this segment or a cost of a travel along this segment.

- OD matrix $(o_{ij})$

Fundamental algorithm supposes that all passengers travel along shortest route from their origins to their destinations.

Fundamental algorithm:

STEP 1 Initialization: For every arc set flow intensity equal to 0, i.e. $\forall a \in A : q(a) = 0$

STEP 2 For ordered pair of nodes $v_i$, $v_j$ of transportation network find the shortest path $m(i, j)$ from $v_i$ to $v_j$.

STEP 3 For every arc $a \in m(i, j)$ set $q(a) = q(a) + o_{ij}$

STEP 4 Repeat STEP 2 and STEP 3 for all ordered pairs of nodes $v_i$ and $v_j$.

Dijkstra's shortest path algorithm can be used in STEP 2 in the form presented in author's paper [3]. The reader can find simple and straightway formulation in Palúch's textbook [7]. Monograph [1] (Cenek, Klima a Janáček) presents an implementation of Dijkstra's algorithm know as Label Set algorithm.

The fundamental algorithm contains the calculation of shortest path between nodes $v_i$, $v_j$ for all ordered pairs $(i, j)$ regardless of the order of these pairs. That is why it is convenient to use suitably modified point to all shortest paths algorithms.

The disadvantage of fundamental algorithm for traffic assignment is that all passengers use always the shortest paths in modeled situation. In real life a passenger has several travelling alternatives – he can choose among them according to his individual preference and actual situation (e.g. available trips, possibility between crowded and comfortable trips, etc.)

Example: A passenger can travel from railway station Vrútky to downtown of Martin by line 10 through urban settlements Košúty and Sever (eastern route) or using something longer line 11 through Záturčie and Podháj (western route), what they really do in real life. But all passengers would use shorter eastern route in fundamental model.

Further shortcoming of fundamental model would appear in the case of too dense transportation network. The fundamental algorithm in such a case would calculate nonzero passenger flows along segments not served by trips. However, this problem can be solved by a reduction of transportation network. Reduced network can be obtained from original one by omitting all arcs not served by trips. If some bus stops remain isolated after network reduction a modification of OD matrix is necessary in the sense that corresponding passengers will travel using nearest not isolated stops.

## 2. Modified algorithm

In the case that there is more alternative routes from origin to destination a passenger can choose among them sometime one and next time other. Palúch presented in [6] a simple and fast $k$ –shortest path algorithm. We can use this algorithm for calculation $k$ shortest alternative paths between nodes $v_i$ and $v_j$ and afterwards to distribute travelling passenger among particular paths.

Modified traffic assignment algorithm:

STEP 1 Initialization: For every arc set flow intensity equal to 0, i.e. $\forall a \in A : q(a) = 0$

STEP 2 For ordered pair of nodes $v_i$ , $v_j$ of transportation network find at most *k* shortest paths $m_1(i, j), m_2(i, j), \ldots, m_k(i, j)$ from $v_i$ to $v_j$ with lengths $d_1, d_2, \ldots, d_k$.

STEP 3 Distribute the amount $o_{ij}$ of number of passengers traveling from $v_i$ to $v_j$ among particular paths $m_1(i, j), m_2(i, j), \ldots, m_k(i, j)$ in proportion $o_{ij}{}^1 : o_{ij}{}^2 : \cdots : o_{ij}{}^k = \frac{1}{d_1} : \frac{1}{d_2} : \cdots : \frac{1}{d_k}$ where $o_{ij}{}^1 + o_{ij}{}^2 + \cdots + o_{ij}{}^k = o_{if}$ .

STEP 4 For every $p = 1, 2, \ldots, k$ and for every $a \in m_p(i, j)$ set $q(a) = q(a) + o_{ij}{}^p$ .

STEP 5 Repeat STEP 2, STEP 3 and STEP 4 for all ordered pairs of nodes $v_i$ and $v_j$ .

This algorithm distribute passengers over more alternative routes in transportation network and is useable in the first phase of design of line in municipal passenger bus transport.

## 3. Experiments

Both algorithm are implemented in C# programming language and used to compute traffic assignment on experimental data taking from electronic tariff system of municipal passenger bus transport of Martin – Vrútky.

Used model of transportation network consists of 113 bus stops, 58 junctions and 380 arcs. Estimated traffics on partial of this transportation network are showed on Figure 1 and Figure 2.

It is easy to see, that modified algorithm distribute flow of passenger better to many segments of transportation network, many of them use alternative path to reach its destination.

In example, only 75 passengers want to travel from “Autoservis Compel” to “Kušúty, sídlisko” in fundamental model on Fig. 1. But this amount is increased to 719 passengers in modified model. This indicates of demand of some bus line from settling Záturčie to Martin downtown throught settling Košúty and Sever. This is not indicated in fundamental model. In real life, this bus line exists.

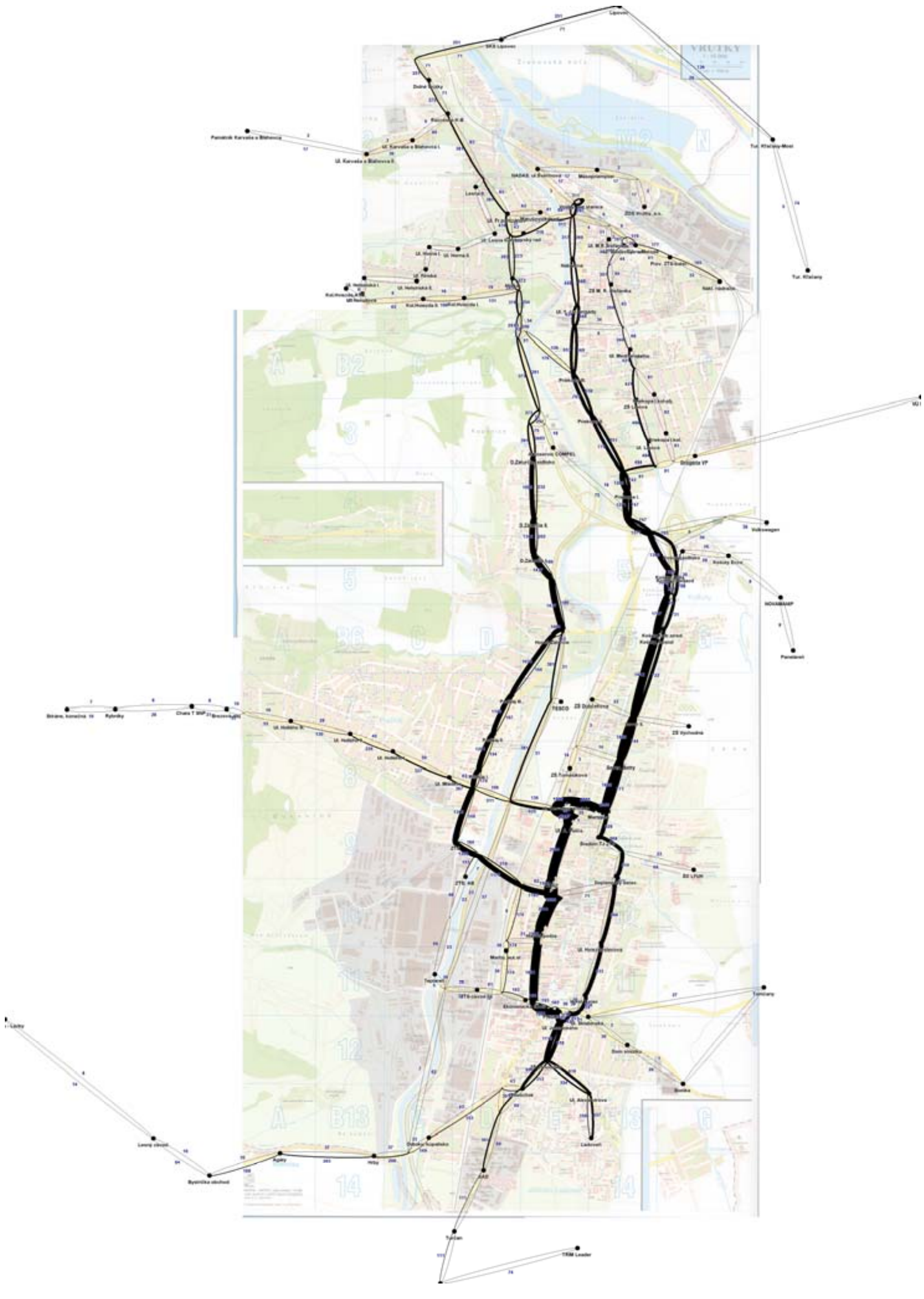

**Figure 1 Flow of passengers in fundamental model**

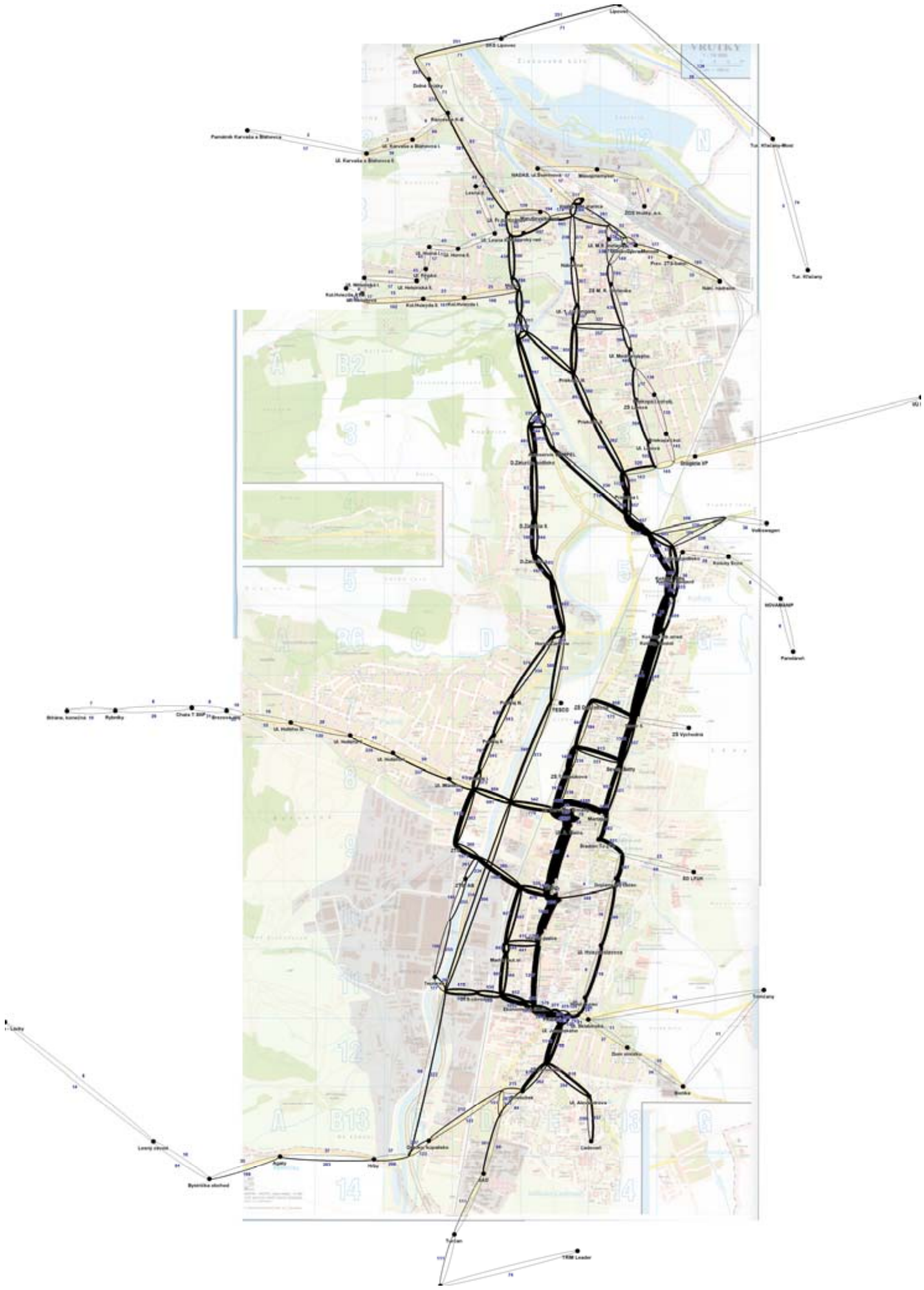

**Figure 2 Flow of passengers in modified model**

## 4. Conclusion

Both mentioned ways of traffic assignment calculate with OD matrix and essential traffic network model containing only arc lengths without making use of it's further parameters. Some authors dealing with traffic assignment take into account additional constraints and objectives like uniformity of traffic loads along network segments. Nevertheless, our attitude was successfully applied in assessment of OD matrix for municipal bus transport of Martin – Vrútky. The result of our attitude was compared to that of complete traffic count in morning peak (5:30 to 8:30) and differences showed to be negligible.

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# DETERMINANTS OF THE FOREIGN DIRECT INVESTMENTS IN THE SLOVAK AND THE CZECH REPUBLIC

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**Abstract:** The aim of this paper is to analyze the relationship between the allocation of Foreign Direct Investment (FDI) in Slovak Republic and Czech Republic and chosen macroeconomic indicators. The aim of the paper is not to analyze the relationship between the Slovak and the Czech Republic; we will only compare the development trends of chosen economic indicators and their effect on FDI. We will further analyse whether there is a relationship between the level of FDI inflows into the countries and the work force supply or the average monthly wage in the countries or the density of roads on $km^2$ country area or the number of persons graduated from secondary vocational schools and universities. We will use year time series from 2002 till 2007. For financial indicators comparison, we will re-count all this indicators on Euro. For the Slovak Republic we use the exchange rate of 30,126 SKK for Euro and for Czech Republic we use the exchange rate 26,465 CZK for Euro (Exchange rate of the Czech national bank as of 31st of December 2007). Correlation analysis will be used for this analysis.
**Keywords:** Foreign Direct Investments, Correlation analysis

## 1 Introduction

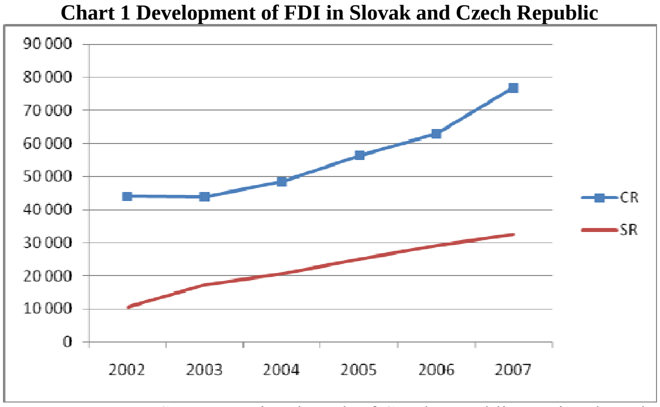
In this paper we study if chosen characteristics determine the amount of FDI inflows in Slovakia and the Czech Republic. The purpose of presented paper is not comparing FDI and its determinant between the Slovak Republic (SR) and the Czech Republic (CR); we want to show the development trends of these determinants on FDI in these two countries. The Slovak Republic and the Czech Republic are very similar countries in language, geographic location, demographic structure and also history. Among these determinates of FDI we choose unemployment in these two countries as a labour supply, average monthly wage in these countries, roading (quality and density of road network) on $km^2$, number of graduate students on high schools and universities. We have data from year 2002 till 2007. As a tool of our analysis we choose correlation analysis between mentioned indicators for Slovakia and the Czech Republic separately. We use also financial indicators, so we transform all of them on Euro currency. Slovak indicators were transformed with exchange rate 30,126 SKK for Euro. For Czech indicators we used exchange rate of Czech National Bank from 31.12.2007, 26,465 CZK for Euro.

## 2 Development of Foreign Direct Investment

International Money Fund (IMF) characterize FDI as a category of international investments that reflects the objective of a resident in one economy (the direct investor) obtaining a lasting interest in an enterprise resident in another economy (the direct investment enterprise). The lasting interest implies the existence of a long-term relationship between the direct investor and the direct investment enterprise, and a significant degree of influence by the investor on the management of the enterprise. A direct investment relationship is established when the direct investor has acquired 10 percent or more of the ordinary shares or voting power of an enterprise abroad.[1]

[1] IMF Balance of Payment Manual, 5th edition, § 359, p. 86

**Chart 1 Development of FDI in Slovak and Czech Republic**




Source: National Bank of Czech Republic, National Bank of Slovak Republic

The FDI volume grew in both countries during the monitored period. Growing of FDI in the Czech Republic was lower than in the Slovak Republic in year 2003. But since year 2006 the growth of FDI in Czech Republic has become very rapid. In summary we can say, that the FDI in the Czech Republic has higher growth rate as in the Slovak Republic.

Foreign investor came to the country throw acquisition of existing company or he brings “Greenfield investments” (f. e. building a new factory). During last years in SR were made some significant Greenfield investments – it was build two car manufacturing companies in year 2003 and 2004.

## 3 Short characteristic of selected indicators

Data base about the Slovak labour market is acquired from the Statistical Office of the Slovak Republic (SOSR) and the Ministry of Labour, Social Affairs and Family. The Statistical Office of SR monitors number of employees according of: quarterly labor force sample survey, monthly statistical survey and also quarterly survey according to ESA95.

The data base about the Czech Republic is acquired from the Czech Statistical Office, according which: The Unemployment comprise all individulas aged 15+ who satisfied **all of the following three conditions** during the reference period:

* were **without work** - i.e., were in neither employment nor self-employment,

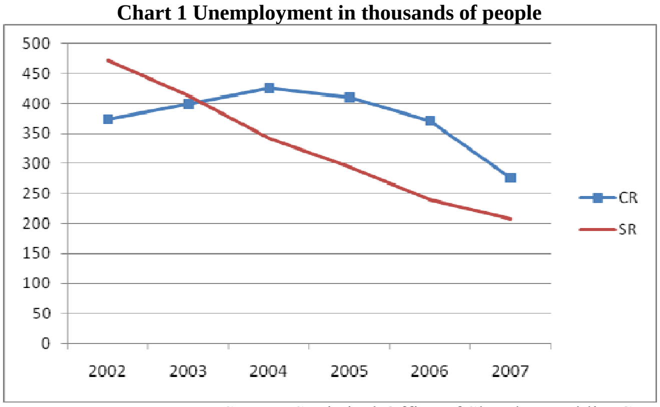
* were **actively seeking for a work**. The active form of seeking work includes registration with a labour office or private employment exchange, checking at work sites, farms, market or other assembly places, placing or answering newspaper advertisements, taking steps to establish own business, applying for permits and licenses, or looking for a job in a different manner,

* were **currently available to work** - i.e., were available during the reference period for paid employment or self-employment immediately or within 14 days.

If the individuals **fail to meet even one of the conditions** above, they are classified **as employed or economically inactive**[2]. The only exceptions are individulas who do not seek for a job because they have found it already but their work will commence later (not later than 3 months). These individuals are also classified to the unemployed by Eurostat definition, so we will use it in our next analyse.

[2] http://www.czso.cz/csu/2010edicniplan.nsf/engt/070025274D/$File/311510m02%20en.pdf

**Chart 1 Unemployment in thousands of people**

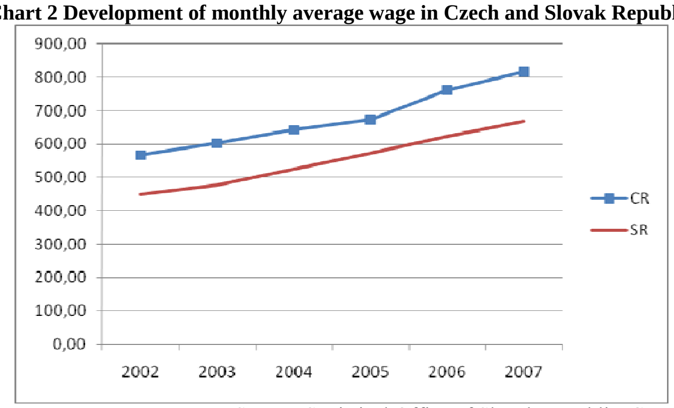



Source: Statistical Office of Slovak Republic, Czech Statistical Office

Unemployment development in both countries has a decreasing trend during the monitored period but its dynamic is slowing down. We can see increase of unemployment during the years 2002 – 2004. Similar growth we can assume in both countries in years 2008 and 2009 because of world economic crisis. We can see a growing unemployment trend line in the Slovak Republic in year 2007.

Chart 3 shows the development of monthly average wage in both countries. With this indicator we would like to show manpower costs for foreign investors. In SR flows FDI into manufacturing corporations, which is in CR in second place after service establishment.

**Chart 2 Development of monthly average wage in Czech and Slovak Republic**

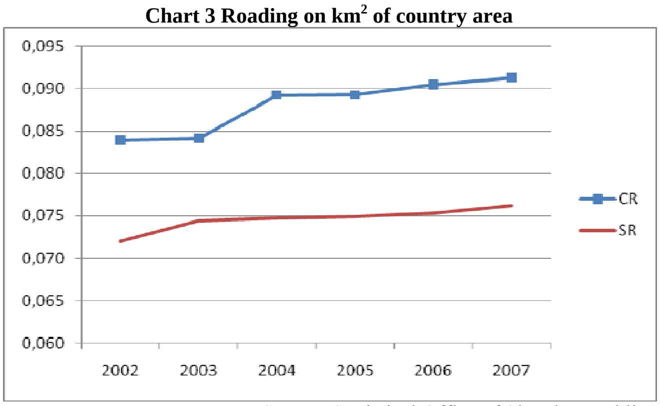



Source: Statistical Office of Slovak Republic, Czech Statistical Office

Average monthly wage in both countries grew, but in CR is approximately about 100 € higher. In SR is the highest average salary in the Bratislava region. This is because of geographic position of the region and also because this is the capital of the country. The other regions fall behind Bratislava region. The lowest average wage is in Prešov region.[3]

Roading could influence the decision of investor about investment in the country in considerable way. For this purpose we use roading on km$^2$ of country area. We assume that for foreign investors upper class roads will be interesting. Therefore in our analyse we use highways, highways feeder, roads of first class and speed roads on km$^2$ of country area.

**Chart 3 Roading on km$^2$ of country area**




Source: Statistical Office of Slovak Republic, Dalnice–silnice.cz[4]

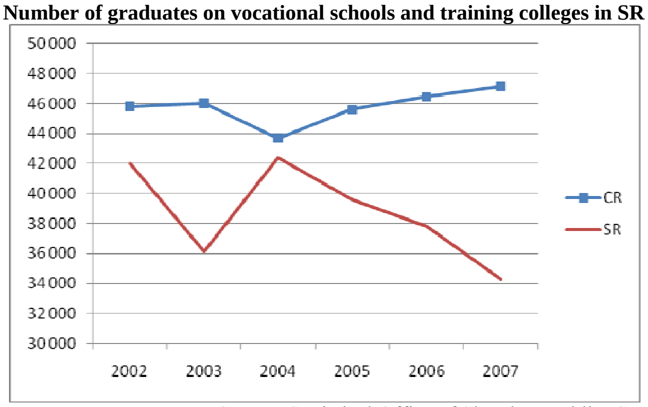
We can see that the Czech Republic is better of in roading. Extensive growth took place in year 2004 in CR. Slow growth was seen in year 2003 also in the Slovak Republic. We except some higher growth in year 2010, when will be opened some highways in middle and eastern Slovakia.

Number of graduates from vocational schools, training colleges and universities shows age composition in the country and also quality of labour. We leave out from our analyze high school graduates, because we assume, that they will continue to study on some university. We can show development of these indicators on next charts.

[3] Domonkos, Mikušová 2009

[4] http://dalnice-silnice.cz/CZ. htm

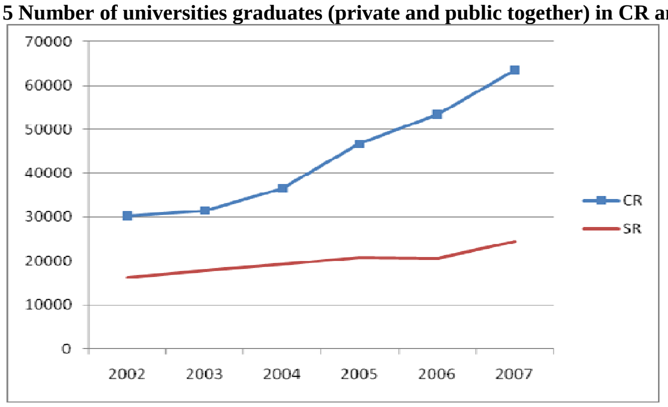
**Chart 4 Number of graduates on vocational schools and training colleges in SR and CR**




Source: Statistical Office of Slovak Republic, Czech Statistical Office

Development of graduate student number (on vocational schools and training colleges) is very different in both countries. In Slovakia it has downtrend – pupils prefer high schools and later they attend universities. The high school is percepted just as a preparation to later university studies. In CR this development is completely different. After year 2004 number of graduate students started to grow and this tendency it has till year 2007.

**Chart 5 Number of universities graduates (private and public together) in CR and SR**




Source: Statistical Office of Slovak Republic, Czech Statistical Office

Number of universities graduates grew during monitored period in both countries. Growth rate is marked in year 2006 in Slovakia.

## 4 Correlation Analysis

In this part of the paper we show the results from correlation analysis between FDI and unemployment development, average monthly wage, roading and education in the Slovak and the Czech Republic.

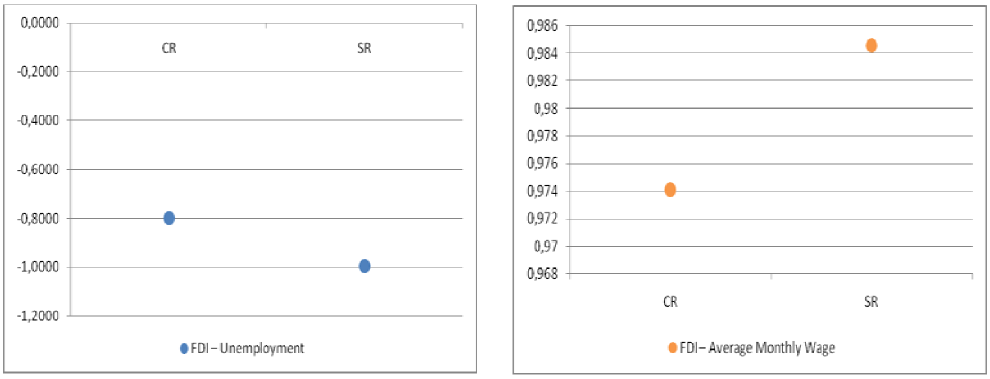
Correlation is any of a broad class of statistical relationships between two or more random variables or observed data values. The most common of these is the Pearson correlation coefficient (correlation coefficient), is mainly sensitive to a linear relationship between the two variables. The correlation coefficient is +1 in the case of a perfect positive (increasing) linear relationship, −1 in the case of a perfect decreasing (negative) linear relationship, and some value between −1 and 1 in all other cases, indicating the degree of linear dependence between the variables. As it approaches zero there is less of a relationship. The closer the coefficient is to either −1 or 1, the stronger the correlation between the variables. If the variables are independent, Pearson's correlation coefficient is 0, but the converse is not true because the correlation coefficient detects only linear dependencies between two variables.

**Table 1 Correlation Coefficient**

| | Czech Republic | Slovak Republic |
|---|---|---|
| FDI – Unemployment | -0,7982 | -0,9953 |
| **FDI – Average Monthly Wage** | **0,97409** | **0,98456** |
| FDI – Roading | 0,8360 | 0,9326 |
| **FDI - Number of graduates on vocational schools and training colleges** | **0,61379** | **-0,61617** |
| FDI – Number of graduates on universities | 0,9909 | 0,9476 |

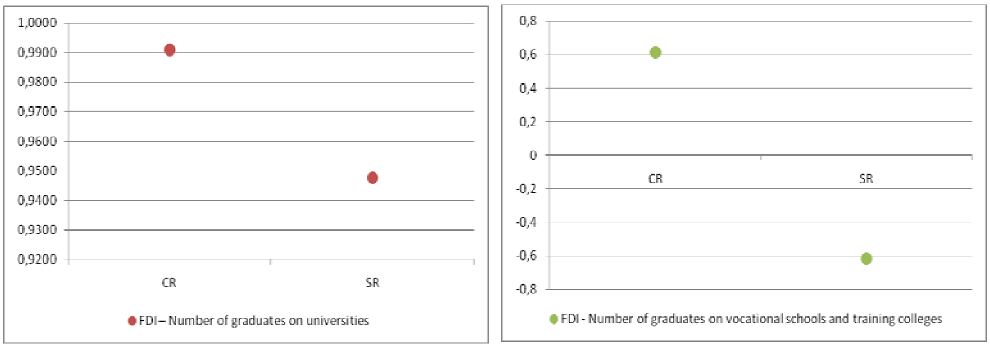
We analyzed if there exists a linear dependence between FDI and chosen indicators. According to indicators we assumed positive or negative linear correlation. We assumed that in the country with high unemployment is enough manpower available. Therefore we assumed positive correlation between FDI and unemployment. This assumption is not fulfilled in the countries. Analysing FDI and average monthly wage we assumed negative correlation, because when the cost of labour is low, the country could be interesting for investors. This hypothesis is also not confirmed. Both countries had negative correlation coefficients.

**Chart 6 Correlation coefficients**



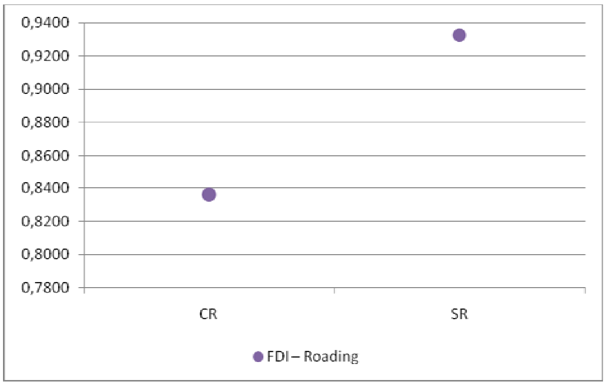

Between FDI and number of graduate students we assumed a positive linear correlation. This correlation is confirmed in CR on all monitored type of schools. In SR is our assumption confirmed only for university graduate students.

**Chart 7 Correlation Coefficients**




Roading influences markedly on FDI in the country. We assumed that dense road infrastructure will make the country more interesting for investors. Therefore we assumed positive correlation between FDI and road infrastructure. This assumption is confirmed for both countries and for SR is the positive correlation much higher than in CR.

**Chart 8 Correlation Coefficients**




## Conclusion

In the presented paper we examined dependences between FDI and chosen macroeconomic indicators. We made the analysis for two countries – the Czech Republic and the Slovak Republic. Financial indicators were transformed to Euro currency. We examined dependences with correlation analysis. We choose labour supply measured with unemployment rate in the country, average monthly wage, roading on $km^2$ of country area, and number of graduate students on vocational schools, training colleges and universities in the country as the macroeconomic indicators. Between FDI and labour supply, roading and number of graduate students we assumed positive linear correlation. Negative linear correlation we assumed between FDI and average monthly wage. Only one from our assumption is confirmed for both countries – number of graduate students on universities. According of this, we can finish our research with conclusion, that FDI in the Czech and the Slovak Republic is not linearly determined by chosen indicators. In the future we will make nonlinear correlation between these indicators. This analysis will bring further interesting conclusions.

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# INVESTMENT OPPORTUNITIES IDENTIFICATION – A MULTIPLE CRITERIA APPROACH

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**Abstract:** The paper presents the developed decision support system for identification of investment opportunities among government bonds through a confrontation of their spreads and level and trends characteristic of the macroeconomic development of corresponding countries. Two concepts of spreads are used. In the first approach the spread is defined as the difference between the yield to maturity of the specific country government bond and Germany government bond with the same maturity. In the second approach the credit default spreads derived from default probability of the country are used. The OECD historical and prognostic macroeconomic data are used for macroeconomic ranking of selected countries. Selected criteria that measures trends and levels are transformed into so called z-scores and application of Promethee methodology provides the final ranking. As a result we have a decision support system in excel environment.

**Keywords:** MCDM techniques, Macroeconomic criteria, Spreads.

## 1. Introduction

There are many applications of known MCDM techniques that as the main and only result offer an outranking of examined variants or alternatives. In many cases a sensitivity analysis on changes in inputs is missing as well. The question about usefulness of such results can arise. In the paper we also use such techniques to rank countries on the base of their macroeconomic characteristics. But this result is confronted with the spreads of the countries with the goal to identify promising investment opportunities among government bonds.

## 2. Model

Suppose we have responses $a_{ij}^t$ of different variants (countries) $i$, $i = 1, \ldots, m$, on different objectives (macroeconomic characteristics) $j$, $j = 1, \ldots, n$, in specific time periods (years) $t$, $t = 1, \ldots, T$. For these responses we define two types of macroeconomic criteria, the criteria of level and the criteria of trend. Let $[d_l, h_l]$, $1 \le d_l \le h_l \le T$, is the time period for level criteria and $[d_r, h_r]$, $1 \le d_r < d_l \le h_l \le h_r \le T$, is the time period for trend criteria, then

$$c_{ij}^l = \frac{\sum_{t=d_l}^{h_l} a_{ij}^t}{h_l - d_l + 1}$$

is the level value of criterion $j$, $j = 1, \ldots, n$, for country $i$, $i = 1, \ldots, m$, and

$$c_{ij}^r = c_{ij}^l - \frac{\sum_{t=d_r}^{h_r} a_{ij}^t}{h_r - d_r + 1}$$

is the trend value of criterion $j$, $j = 1, \ldots, n$, for country $i$, $i = 1, \ldots, m$.

So we have a multiple criteria decision making problem with *m* alternatives and 2*n* criteria and we can use some of known methods, (Mlynarovič, 1998) to rank the alternatives. But in this application we suggest to use so called *z – scores* instead of level and trend values of criteria. The advantageous of such approach is the fact that in this case we take into account not only the criteria values, but also their variability.

Let $\mu_j^l$ and $\sigma_j^l$ are the mean and the standard deviation of $c_{ij}^l, i = 1,\dots,m$, and $\mu_j^r$ and $\sigma_j^r$ are the mean and the standard deviation of $c_{ij}^r, i = 1,\dots,m$, then

$$s_{ij}^l = \frac{c_{ij}^l - \mu_j^l}{\sigma_j^l}, \quad i = 1,\dots m, \ j = 1,\dots n$$

are $z - scores$ for level criteria and

$$s_{ij}^r = \frac{c_{ij}^r - \mu_j^r}{\sigma_j^r}, \quad i = 1,\dots m, \ j = 1,\dots n$$

are $z - scores$ for trend criteria.

Countries macroeconomic outranking can be written as a multiple criteria decision making problem

$$\text{"max"} \quad \left\{\mathbf{s}_i = \left(\mathbf{s}_i^l, \mathbf{s}_i^r\right) = \left(s_{i1}^l, \dots s_{in}^l, s_{i1}^r, \dots s_{in}^r\right) \quad i = 1,\dots m\right\}$$

where, without loss of universality, it is assumed that “the more the better” is applied for all criteria. Suppose that application of the PROMETHEE II method provides the values of net flows $\Phi_i$, *i* = 1, …, *m*, for which

$$\Phi_1 > \Phi_2 > \dots \Phi_i > \dots \Phi_m$$

It is known that in the PROMETHEE terminology it means that the country 1 is the best one and the country *m* is the worst one.

Let $p_i$, $i = 1, \dots, m$, is the credit default spread for country *i* derived from default probability of the country. It holds the higher value of $p_i$ then higher probability of default for country *i*. Suppose that $r_i$, $i = 1, .., m$, is the yield to maturity of government bonds offered by the country *i*. In the world of ideal information for given PROMETHEE results should by hold

$$p_1 < p_2 < \dots p_i < \dots < p_m$$

$$r_1 < r_2 < \dots r_i < \dots < r_m$$

because the worse country the higher return must be offered. In this situation the potential investor has only to decide how much risk he is willing to accept. In practical situations one can hardly expect such unambiguous result and possible contradictions between macroeconomic results and spreads provide a space for identification good investment opportunities.

## 3. Data

The OECD [1] regularly provides the historical and prognostic yearly macroeconomic data about:

- demand and output,

[1] www. oecd.org

- wages, costs, unemployment and inflation,
- key supply side data,
- saving,
- fiscal balances and public indebtedness,
- interest rates and exchange rates,
- external trade and payments,
- other background data,

These data for the period from 1992 to 2011 were used to construct the following macroeconomic criteria:

1. Gross value added at 2000 basic prices excluding FISIM: total economy (OVGE)
2. Private final consumption expenditure at 2000 prices (OCPH)
3. Gross fixed capital formation at 2000 prices: total economy (OIGT)
4. Unemployment rate: total :- Member States: definition EUROSTAT (ZUTN)
5. Harmonized consumer price index (All-items) (ZCPIH)
6. Saving rate, gross: households and NPISH (Gross saving as percentage of gross Disposable income) (ASGH)
7. National income at current market prices (UVNN)
8. Final consumption expenditure of general government at 2000 prices (OCTG)
9. Budget deficit as % of GDP
10. Implicit interest rate: general government :- Interest as percent of gross public debt of preceding year Excessive deficit procedure (based on ESA 1995) (AYIGD)

11. General Government expenditures as % of GDP
12. Nominal long-term interest rates (ILN)
13. Total exports of goods :- Foreign trade statistics (DXGT)
14. Total imports of goods :- Foreign trade statistics (DMGT)
15. Current account balances as a percentage of GDP
16. Export market growth in goods and services

Macroeconomic results derived from these data are finally confronted with current spreads[2] presented at the Table 1. As the spread are calculated fro, bonds (depth instrument of the country) of different countries, in terms of ranking we put more relative importance (higher weight) to the depth macroeconomic data.

## 4. The decision support system and its application

The decision support system developed in excel environment realizes Promethee II outranking method including sensitivity analysis for the importance weights for the selected criteria. The user can select:

- the period for level criteria,
- the period for trend criteria,
- countries from the list,
- criteria from the list and assign the importance weights
- control parameters for Promethee II method.

[2] Bloomberg, 2010

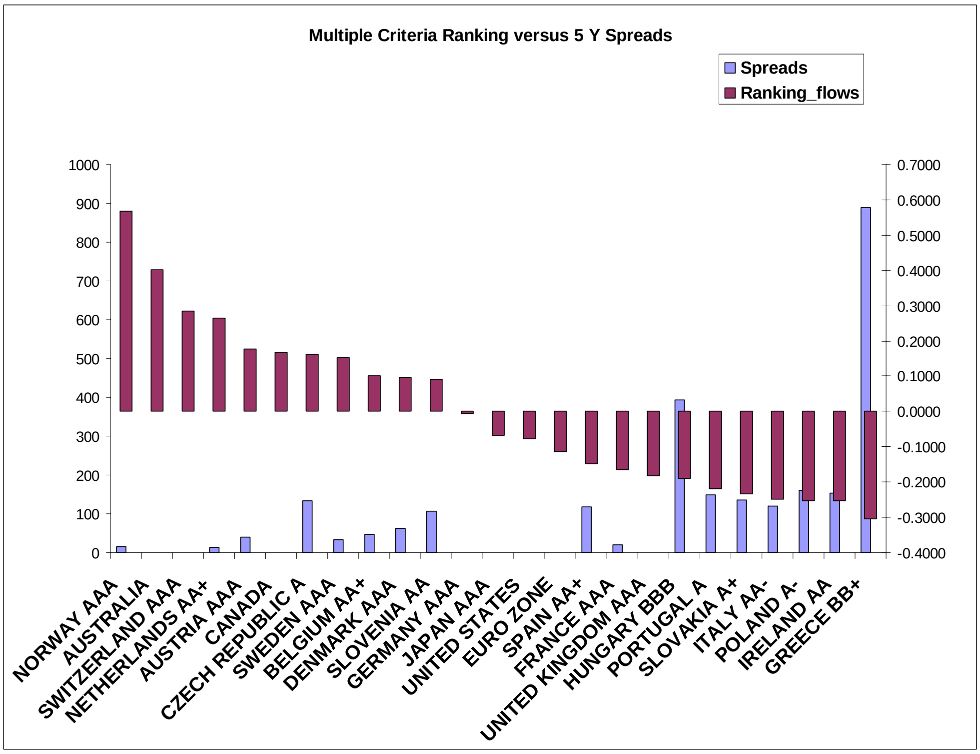


Figure 1: Results for spreads owing to Germany bonds

Table 1: Spreads for selected countries

| Country | Spread | | Country | Spread | |
|---|---|---|---|---|---|
| | Credit Default | via Germany Bond | | Credit Default | via Germany Bond |
| Austria | 80.3 | 39.41 | Italy | 166 | 119.38 |
| Belgium | 128.2 | 45.72 | Latvia | n.a | n.a. |
| Bulgaria | 310.1 | n.a. | Lithuania | 266.5 | n.a. |
| Cyprus | n.a. | n.a. | Luxembourg | n.a. | n.a. |
| Czech Republic | 96 | 132.9 | Netherlands | 45.9 | 14.34 |
| Denmark | 40.4 | 61.91 | Poland | 134.4 | 159.74 |
| Estonia | 110 | n.a. | Portugal | 291.9 | 148.87 |
| Finland | 31.1 | 11.25 | Romania | n.a. | n.a. |
| France | 78.6 | 19.49 | Slovakia | 87.7 | 134.9 |
| Germany | 37.3 | 0 | Slovenia | 79.1 | 107.31 |
| Greece | 807.9 | n.a. | Spain | 216.3 | 117.42 |
| Hungary | 302.3 | n.a. | Sweden | 37.9 | 32.4 |
| Ireland | 242.4 | 153.15 | United Kingdom | 78.9 | n.a. |

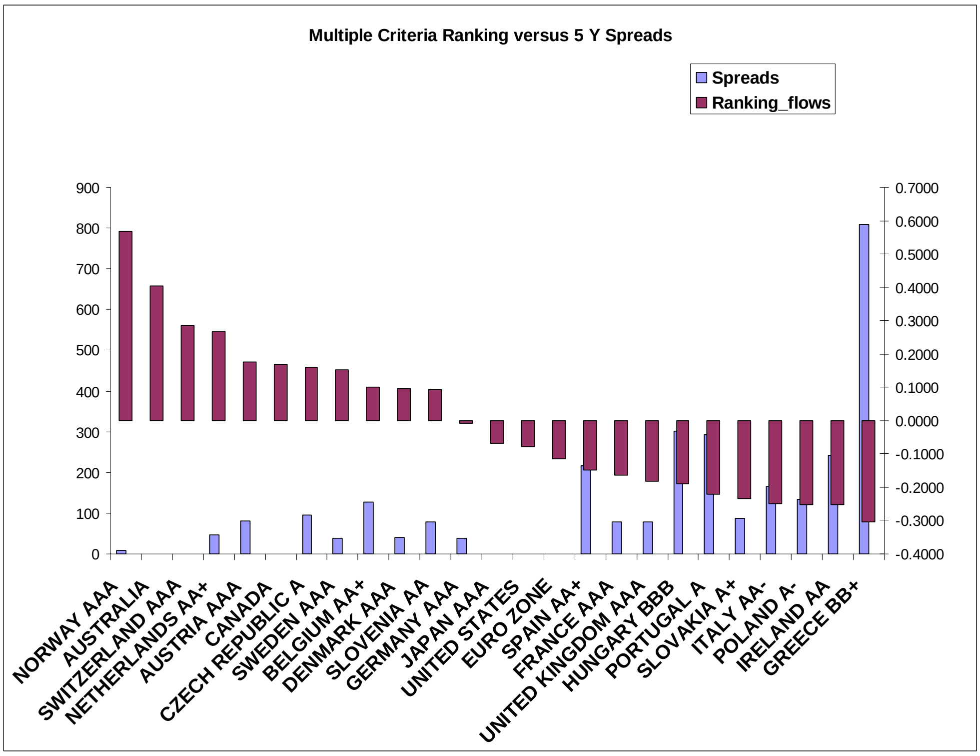


Figure 2: Results for credit default spreads

In Figure 1 there is a basic result of the confrontation of macroeconomic results, where for level criteria the period from 2010 to 2011 was selected and for the trend criteria the period from 200 to 2009 was selected. In this illustration the spreads owing to 5 years Germany bond were selected. In figure 2 there are corresponding results for credit default spreads.

## 5. Conclusions

From the macroeconomic developments off EU countries during the last year we could see that the ranking model has a good ability to indentify countries with worst and best economic conditions. Among the best ranked countries there are Norway, Australia, Switzerland and Austria. Results in figures help us to answer in which countries we wouldn't invest in any case. The results shown in figures 1 and 2 help investors to identify in which countries to invest (like Norway, Australia, Switzerland, Austria), which are ranked among the best and do have the highest current spread (defined as spread vs. German bunds) comparing to others top ranked countries.

From the graphs we also could see that values of credit default swaps spreads are not the same as spreads from cash bonds (difference country bond vs. German bund with same maturity). The reason is the liquidity (not all cash bonds are liquid) and credit default swap reflects probability of default each country (that means how much one need to pay extra vs. risk free to protect against a default of chosen country). On the other hand cash bond spreads are simply the spreads which one pays if he wants to invest in particular bonds.

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# LARGEST CLIQUES AND DECOMPOSITION OF BUS SCHEDULING PROBLEM

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**Abstract:** This paper shows several ways of exact and heuristic decomposition of bus scheduling problem. We show that largest cliques in trip digraph have important role for exact decomposition.

**Keywords:** Vehicle scheduling, largest independent set, decomposition.

## 1. Introduction.

Vehicle scheduling problem in regular regional and/or municipal personal bus transport has an important role in economics of any bus provider. An optimal bus schedule minimizes the number of buses and the total cost without any negative consequences to travelling passengers. Corresponding mathematical models belong to NP-hard discrete optimization problems where decomposition can help to achieve a good suboptimal solution. Vašek [3] was the first to find out the importance of largest cliques for exact decomposition of bus scheduling problem, Engelthaller [1] and Peško [2] applied cliques in heuristic minimization of vehicle fleet with flexible trips.

### 1.1. Fundamental notions

*Trip* is a travel from a starting point to a finishing point of a route and it is considered to be an elementary amount of the work of a bus. Suppose we are given a set of trips $S = \{s_1, s_2, \dots, s_n\}$. We will say that the trip $s_i$

*precedes* the trip $s_j$ and we will write $s_i \prec s_j$ if the same bus can provide first the trip $s_i$ and afterwards the trip $s_j$. Relation $\prec$ is antireflexive and transitive on *S*. *Running board* of a bus is an arbitrary nonempty sequence $T = (s_1, s_2, \ldots, s_m)$ of trips with property $s_1 \prec s_2 \prec \ldots \prec s_m$. The *cost* of running board is defined as

$$c(T) = \sum_{i=1}^{m-1} c(s_i, s_{i+1}), \qquad (1)$$

where $c(s_i, s_{i+1})$ is a linkage cost of corresponding trips. Linkage cost includes dead mileage expenses, however, it may include waiting costs, line changing penalties and other penalties as well. *Bus schedule* of the set *S* of trips is a set of running boards $O = \{T_1, T_2, \ldots, T_k\}$ such that every trip of the set *S* occurs exactly in one running board of *O*. The *total cost of bus schedule O* is

$$c(O) = \sum_{T \in O} c(T). \qquad (2)$$

The cost $c(O)$ of a bus schedule fulfilling (1) with $c(T)$ defined in (2) is called *linear*.

The fundamental vehicle scheduling problem **– FVSP –** is to find a bus schedule with the minimum number of running boards and with the minimum total cost.

### 1.2. Algorithms for Fundamental Vehicle Scheduling Problem

Several mathematical models are used to solve the fundamental vehicle scheduling problem. Integer linear programming model solves the FVSP as an assignment problem. Network flow model formulates FVSP as a max flow min cost model. We will use the following graph model in this paper.

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of trips with precedence relation $\prec$. Trip digraph is a digraph $G_S = (V, E, c)$, where $V = S$ and where $(s_i, s_j) \in E$ if and only if $s_i \prec s_j$ . The edge weight $c(s_i, s_j)$ is the cost of $(s_i, s_j)$. It follows from properties of relation $\prec$ that $G_S$ is an acyclic and transitive digraph.

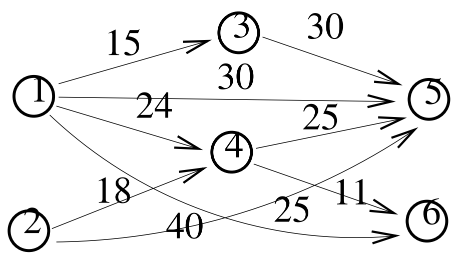


Fig. 1. Trip digraph.

Arbitrary path in $G_S$ represents a feasible running board and vice verse – every running board can be represented as a path in $G_S$ . So the fundamental vehicle scheduling problem can be formulated as follows: To find a disjoint path covering $G_S$ with minimum cardinality and with minimum total length. Just formulated problem can be solved as to find a flow with minimum value and minimum cost covering all vertices of a network.

All mentioned formulations lead to polynomial problems which have very good and fast algorithms. However, practical requirements are formulated in more complex running board cost – e. g. if we want that all vehicles return to their starting places, the cost of running board changes to $c'(T) = \sum_{i=1}^{m-1} c(s_i, s_{i+1}) + c(s_m, s_1)$ and the cost of bus schedule becomes

$$c'(O) = \sum_{T \in O} c'(T). \qquad (3)$$

Vehicle scheduling problem with this cost is no longer polynomial. Then a suitable decomposition of the problem can help good suboptimal solution.

The cost $c(O)$ of a bus schedule fulfilling (3) where $c'(T)$ is an arbitrary objective function is called *separable*.

## 2. Independent Sets in Trip Digraph and Vertical Decomposition for a Linear Objective Function.

Let $G_S = (V, E, c)$ be a trip digraph, let $W \subseteq V$ be a set o trips. *W* is called *independent set* if for any couple of trips $s_i, s_j \in W$ does not hold $s_i \prec s_j$ nor $s_j \prec s_i$. *W* is a *maximum independent set* or a *clique* if there does not exist an independent set *W'* different from *W* such that $W \subset W' \subseteq V$. *W* is a *largest independent set* or a *largest clique* if there does not exist an independent set *W'* such that $|W| < |W'|$, i.e. if *W* is an independent set with maximum cardinality.

The following theorem holds:

**Theorem**. Dilworth. *Let G be an acyclic transitive digraph. Then the minimum number of paths covering all vertices is equal to the cardinality of largest independent set in G.*

Since our trip digraph $G_S = (V, E, c)$ is acyclic and transitive, it follows from Dilworth's theorem that minimum number of running boards is equal to the cardinality of largest independent set $W = \{w_1, w_2 \dots, w_k\}$. Let $O = \{T_1, T_2, \dots, T_k\}$ be a bus schedule with minimum number of running boards. Then $k = |W|$ and every trip from *W* has to be exactly in one running board from *O*.

We can define two sets of vertices:

$V_1 = \{s \mid s \in V,\ s \prec w_i \ \textit{for some}\ w_i \in W\}$,

$V_2 = \{s \mid s \in V,\ w_j \prec s \ \textit{for some}\ w_j \in W\}$

If $s \in V_1 \cap V_2$ then $w_j \prec s$ and $s \prec w_i$ for some $w_i, w_j \in W$. From transitivity of relation $\prec$ it follows that $w_j \prec w_i$. If $w_j = w_i$ we have contradiction with antireflexivity of $\prec$, if $w_j \neq w_i$ we have contradiction with the fact that $W$ is a independent set. Hence it holds $V_1 \cap V_2 = \{\ \}$.

Now we can decompose the original set $S$ of trips into two sets $S_1 = V_1 \cup W$ and $S_2 = V_2 \cup W$. Let the following solutions are optimal solution for $S_1$, $S_2$ for a linear objective function (2) :

$$T_{11} = s_{11} \prec s_{12} \prec \cdots \prec s_{1p_1-1} \prec s_{1p_1} = w_1$$
$$T_{12} = s_{21} \prec s_{22} \prec \cdots \prec s_{2p_2-1} \prec s_{2p_2} = w_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T_{1k} = s_{k1} \prec s_{k2} \prec \cdots \prec s_{kp_k-1} \prec s_{kp_k} = w_k$$
$$T_2 = s_{1p_1} = w_1 \prec s_{1p_1+1} \prec \cdots \prec s_{1m_1}$$
$$T_{22} = s_{2p_2} = w_2 \prec s_{2p_2+1} \prec \cdots \prec s_{2m_2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T_{2k} = s_{kp_k} = w_k \prec s_{kp_k+1} \prec \cdots \prec s_{km_k}$$

Then the following set of running boards is an optimum solution for whole set $S$ :

$$T_1 = s_{11} \prec s_{12} \prec \cdots \prec s_{1p_1-1} \prec s_{1p_1} = w_1 \prec s_{1p_1+1} \prec \cdots \prec s_{1m_1}$$
$$T_2 = s_{21} \prec s_{22} \prec \cdots \prec s_{2p_2-1} \prec s_{2p_2} = w_2 \prec s_{2p_2+1} \prec \cdots \prec s_{2m_2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T_k = s_{k1} \prec s_{k2} \prec \cdots \prec s_{kp_k-1} \prec s_{kp_k} = w_k \prec s_{kp_k+1} \prec \cdots \prec s_{km_k}$$

As well known, the problem of finding a largest independent set in a general graph or digraph is NP—hard. We have showed that there exists a polynomial procedure for the special case of trip digraph. Unfortunately, explanation of this procedure would exceed the range of this paper.

## 3. Horizontal and Vertical Decompositions for a Separable Objective Function.

Running boards of vehicles are closely associated with running boards of corresponding drivers in personal bus transport in Slovakia and Czech Republic. One running board of a bus is performed by one driver or by two drivers. Therefore every running board has to comply with all safety standards and all requirements of Labor Code as safety break, meal break, duration of driver shift etc. Most of mentioned requirements can be modeled as vehicle scheduling problem with a complicated nonlinear but separable objective function. Resulting mathematical problem is NP—hard and therefore a suboptimal heuristic optimization procedure can only be used.

### 3.1 Horizontal Decomposition

Minimization of the number of vehicles is the most important condition of vehicle scheduling process. Therefore we start in this procedure with a starting bus schedule $O = \{T_1, T_2, \ldots, T_k\}$ with minimum cardinality and generally with arbitrary cost. However practical computations showed that better results can be obtained with the cost $c(O) = \sum_{T \in O} c(T)$ expressing total linkage cost.

The second step of this decomposition is to create smaller instances $I_1 = \{T_{11}, T_{12}, \ldots, T_{1k_i}\}$, $I_2 = \{T_{21}, T_{22}, \ldots, T_{2k_2}\}$, ..., $I_r = \{T_{r1}, T_{r2}, \ldots, T_{rk_r}\}$ and to solve vehicle scheduling with complex cost separately for trips from $I_1, I_2, \ldots, I_r$.

Procedures improving complex cost showed to be very fruitful. After optimizing partial schedules for $I_1, I_2, \ldots, I_r$ another decomposition can be tried.

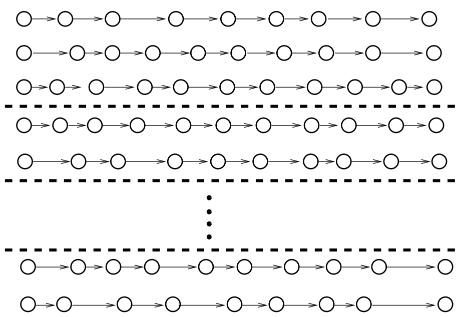

Fig. 2. Horizontal decomposition

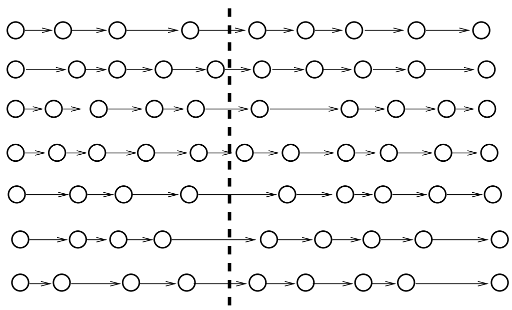

Fig. 3. Vertical decomposition

### 3.1 Vertical Decomposition

Vertical decomposition also starts with a bus schedule

$$T_1 = s_{11} \prec s_{12} \prec \cdots \prec s_{1p_1-1} \prec s_{1p_1} \quad \prec s_{1p_1+1} \prec \cdots \prec s_{1m_1-1} \prec s_{1m_1}$$
$$T_2 = s_{21} \prec s_{22} \prec \cdots \prec s_{2p_2-1} \prec s_{2p_2} \quad \prec s_{2p_2+1} \prec \cdots \prec s_{2m_2-1} \prec s_{2m_2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T_k = s_{k1} \prec s_{k2} \prec \cdots \prec s_{kp_k-1} \prec s_{kp_k} \quad \prec s_{kp_k+1} \prec \cdots \prec s_{km_k-1} \prec s_{km_k}$$

having minimum cardinality and a minimum linear cost.

The part $Head(T_i) = s_{i1} \prec s_{i2} \prec \cdots \prec s_{ip_i-1} \prec s_{ip_i}$ is called the *head of running board* $T_i$ the part $Tail(T_i) = s_{ip_i+1} \prec \cdots \prec s_{im_i-1} \prec s_{im_i}$ is called the *tail* of $T_i$. For every ordered pair *i, j* such that $s_{ip_i} \prec s_{jp_j+1}$ we can compute the complex cost $D_{ij}$ of running board $Head(T_i) \prec Tail(T_j)$. If $s_{ip_i}, s_{jp_j+1}$ are not linkable we set $D_{ij} = \infty$ (resp. $D_{ij} = M$, where $M$ is a large prohibitive constant). We face now to an assignment problem – to assign heads to tails in order to minimize the total cost. The question arises where to divide current running boards into heads and tails – how to determine trips $s_{1p_1}, s_{2p_2}, \ldots s_{kp_k}$. We used to split running boards by departure time *t* – if a trip departs before *t* it belongs to the head of running board otherwise it belongs to the tail. We have designed a procedure which checks all possible decompositions by the sequence of time instants gained as an ordered sequence of departure and arrival time of all trips.

Significant improvement of this attitude was achieved by splitting every running board into three parts – head, body and tail. Now we can fix head and tail of all running board and try to optimally assign bodies.

Just described procedures were implemented in vehicle and crew optimization system KASTOR and were successfully used in municipal and regional bus transport optimization in several tens Czech a Slovak towns.

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# THE USE OF MULTICRITERIA PROGRAMMING TO THE MAINTENANCE OF A MINIMUM NUMBER OF ROAD TRAFFIC[1]

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**Abstract:** In the paper we consider an approach for solving the modification of minimum spanning tree problem using the methods of multicriteria programming. The goal of minimum spanning tree problem is to find an optimal connection within *n* nodes of a connected weighted graph G, so that there is a connection between all nodes with the use of a set of edges so that the sum of weights of the used edges is as minimal as possible. Further on, we will construct a system with a minimal value of used edges providing connection of all system components with the modification to ensure a minimum value of the total distance traveled from the center to each node in case of individual transport. The principle lies in separate problem solution by the choice of several potential centers.

**Keywords:** Minimum Spanning Tree Problem, Location, Goal Programming.

## 1. Minimum spanning tree problem

In minimum spanning tree problem we assume connected weighted graph with *n* nodes and edges. Every edge is evaluated with weight $c_{ij}$ ($i$ =

[1] This paper is supported by the Grant Agency of Slovak Republic – VEGA, grant no. 1/0360/10 „MODEL OF DESIGN NETWORK OF COLLECTING POINT FOR RECYCLING IN THE SLOVAK REPUBLIC“.

1, 2,...$n$, $j$ = 1, 2,...$n$). The goal is to find minimum spanning tree so that the sum of used edges is minimal (minimal lenght of used route). The use of egde (route) from $i$-th do $j$-th node represents binary variable $x_{ij}$ ($i$ = 1, 2,...$n$, $j$ = 1, 2,...$n$) that is equal to 1. Objective function $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$, where $c_{ij}$ is a distance between $i$-th a $j$-th node, represents total traveled distance (sum of weights of used edges) so that:

$$c_{ij} = \begin{cases} c_{ij}, & \text{If there exist the edge betveen } i\text{-th and } j\text{-th node,} \\ 0, & \text{If } i = j, \\ +\infty, & \text{If there exist no edge betveen } i\text{-th and } j\text{-th node.} \end{cases}$$

*Problem formulation*

The above mentioned problem can be formulated as a mathematical programming problem, where binary variables $x_{ij} \in \{0,1\}$, for $i, j = 1, 2, ... n$, are used. The relevant variable will be equal to 1, if the edge between $i$-th a $j$-th node is a part of system of used roads or the variable is equal to 0 otherwise. The values of other used variables $y_{ij}$, for $i, j = 1, 2, ... n$, are limited by interval $\langle 0; (n-1) \rangle$, so that they represent the number of using of relevant route and also they preserve there is a connected route from relevant node to the initial node.

Then, the mathematical formulation of minimum spanning tree problem is:

$$f(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \rightarrow \min$$

$$\sum_{j=2}^{n} x_{1j} = 0$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 2,3,...n$$

$$\sum_{j=1}^{n} y_{ij} - \sum_{j=2}^{n} y_{ji} = 1, \qquad i = 2,3,...n-1$$

$$0 \le y_{ij} \le (n-1)x_{ij}, \qquad i, j = 1,2,...n$$

$$x_{ij} \in \{0,1\}, \qquad i, j = 1,2,...n$$

## 2. Problem modification

In practical application of before mentioned mathematical models may be a problem of selection of a node, which is near the selected center and in the case a lower distance other than the selected center just select the path to that node.

That problem can be solved with the use of second objective function, which in addition to the objective of maintaining a minimum road network will also provide a minimum distance in the case of private transport from the center to each node. Then, the minimum spanning tree problem includes also variables $y_{ij}$ that represent how many times the road between *i*-th and *j*-th node is used. The objective $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}y_{ij}$ represents the total traveled distance from center to individual nodes. Then, multicriteria programming problem can be formulated in the form:

$$f_1(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} \rightarrow \min$$

$$f_2(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} y_{ij} \rightarrow \min$$

$$\sum_{j=2}^{n} x_{1j} = 0$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 2,3,\dots n$$

$$\sum_{j=1}^{n} y_{ij} - \sum_{j=2}^{n} y_{ji} = 1, \qquad i = 2,3,\dots n-1$$

$$0 \le y_{ij} \le (n-1) x_{ij}, \qquad i, j = 1,2,\dots n$$

$$x_{ij} \in \{0,1\}, \qquad i, j = 1,2,\dots n$$

So that problem can be solved as goal programming problem. The design of the goal programming assumes the type of objective function is to maximize. So it is necessary to adapt the multicriteria programming problem to the desired shape. To solve the problem it is needed to establish the weights $\lambda_1 \ge 0, \lambda_2 \ge 0$ for both criteria. One way of weighting is to determine the lowest ($y_1^0$ and $y_2^0$) and highest ($y_1^1$ and $y_2^1$) values for each criterion and on the base of relation

$$\frac{\lambda_2}{\lambda_1} = \frac{y_1^1 - y_1^0}{y_2^1 - y_2^0}$$

it is possible to calculate the weights. Further on, it is necessary to establish a set of targets for both criteria, or to consider the separate optimal solutions for individual objective as the target and to solve the problem by using the percentage deviations.

Using the $L_1$-metric indicated the formulation of linear programming problem as follows:

$$f\left(\mathbf{x},\mathbf{y},\mathbf{o}\right)=\lambda_1 o_1^+ + \lambda_2 o_2^+ \to \min$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - y_1^0 o_1^+ \quad \le \quad y_1^0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}y_{ij} - \quad y_2^0 o_2^+ \le \quad y_2^0$$

$$\sum_{j=2}^{n} x_{1j} = 0$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 2,3,...n$$

$$\sum_{j=1}^{n} y_{ij} - \sum_{j=2}^{n} y_{ji} = 1, \qquad i = 2,3,...n-1$$

$$0 \le y_{ij} \le \left(n-1\right)x_{ij}, \qquad i,j = 1,2,...n$$

$$x_{ij} \in \left\{0,1\right\}, \qquad i,j = 1,2,...n$$

$$o_1^+, o_2^+ \ge 0$$

Using $L_\infty$-metric refers to the linear programming problem:

$$f\left(\mathbf{x},\mathbf{y},\alpha\right)=\alpha \to \min$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \quad - \frac{y_1^0}{\lambda_1}\alpha \le \quad y_1^0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}y_{ij} - \frac{y_2^0}{\lambda_2}\alpha \quad \le \quad y_2^0$$

$$\sum_{j=2}^{n} x_{1j} = 0$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 2,3,...n$$

$$\sum_{j=1}^{n} y_{ij} - \sum_{j=2}^{n} y_{ji} = 1, \qquad i = 2,3,...n-1$$

$$0 \le y_{ij} \le \left(n-1\right)x_{ij}, \qquad i,j = 1,2,...n$$

$$x_{ij} \in \left\{0,1\right\}, \qquad i,j = 1,2,...n$$

$$\alpha \ge 0$$

## 3. Solution by GAMS

Source code for GAMS for solving the goal programming using the $L_1$-metric is the following:

```
Sets
  i vystup /1*n/
  subi1(i) /1/
  subi2(i) /2*n/
  alias (i,j)
  alias (i,k)
  alias (subi2,subj2);
  Scalar   l1 /l1/
  l2 /l2/
  c1 /c1/
  c2 /c2/;
 Table c(i,j);
 Variables x(i,j)
 z
 y(i,j)
 o1
 o2
 uc;
 Binary Variable x;
 Positive Variable y
 o1
 o2;
 Equations
 prve(subi1)
 prve2(subi2)
 druhe(subi2)
 tretie(i,j)
 piate
 sieste
 ucel;
 ucel.. uc=e=l1*o1+l2*o2;
 piate.. sum((i,j),c(i,j)*x(i,j))-c1*o1=l=c1;
 sieste.. sum((i,j),c(i,j)*y(i,j))-c2*o2=l=c2;
 prve(subi1(i)).. sum(j,x(i,j))=e=0;
 prve2(subi2(i)).. sum(j,x(i,j))=e=1;
 druhe(subi2(i)).. sum(j,y(i,j))-sum(subj2(k),y(k,i))=e=1;
 tretie(i,j).. y(i,j)-(n-1)*x(i,j)=l=0;
 Model CPkostraL1 /all/;
 Solve CPkostraL1 using mip minimizing uc;
 Display x.l;
```

Source code for GAMS for solving this problem using the $L_\infty$-metric is as follows:

```
Sets  i vystup /1*n/
   subi1(i) /1/
   subi2(i) /2*n/
   alias (i,j)
   alias (i,k)
   alias (subi2,subj2);
   Scalar  l1 /l1/
   l2 /l2/
   c1 /c1/
   c2 /c2/;
   Table c(i,j);
  Variables x(i,j)
  z
  y(i,j)
  alfa
  uc;
  Binary Variable x;
  Positive Variable y
  alfa;
  Equations    prve(subi1)
  prve2(subi2)
  druhe(subi2)
  tretie(i,j)
  piate
  sieste
  ucel;
  ucel.. uc=e=alfa;
  piate.. sum((i,j),c(i,j)*x(i,j))-c1*alfa/l1=l=c1;
  sieste.. sum((i,j),c(i,j)*y(i,j))-c2*alfa/l2=l=c2;
  prve(subi1(i)).. sum(j,x(i,j))=e=0;
  prve2(subi2(i)).. sum(j,x(i,j))=e=1;
  druhe(subi2(i)).. sum(j,y(i,j))-sum(subj2(k),y(k,i))=e=1;
  tretie(i,j).. y(i,j)-(n-1)*x(i,j)=l=0;
  Model CPkostraLnek /all/;
  Solve CPkostraLnek using mip minimizing uc;
  Display x.l;
```

**Conclusions**

The adduced paper presents the utilization possibilities of the mathematical programming task in different conditions and goals of chosen location problem as well as their solutions by using the software product GAMS.

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# MAX-ALGEBRA FOR BUS LINE SYNCHRONIZATION

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**Abstract:** Max-algebra is an attractive way to describe a class of nonlinear problems that appear for instance in discrete event dynamic systems. This paper focuses on modeling bus line timetables on bus transportation network. We discuss application corresponding max-algebraic eigenvalues and eigenvectors to enable passengers to change bus lines.

**Keywords:** Max-algebra, eigenvalues, eigenvectors, bus line timetables

## 1. Motivation

Max-algebra is an analogue for classical linear algebra, developed in the 1960s to study certain industrial production, transportation and related discrete event dynamic systems [1, 5].

We focus on modeling bus line timetables on a bus transportation network. We are interested in depature time of buses from the stops. The modeling issues in this paper will be ilustrated by a simple network for three bus lines given in Figure 1.

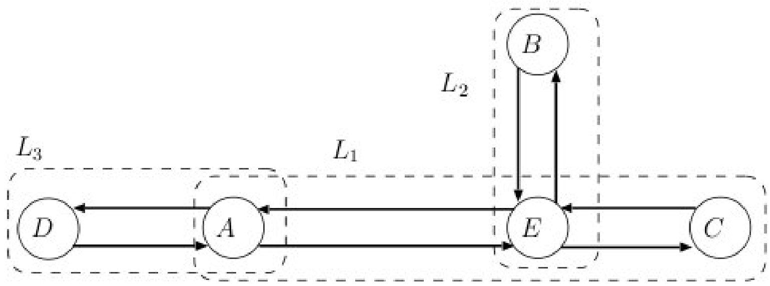


Figure 1: A simple bus network for lines $L_1$, $L_2$ and $L_3$

There are line $L_1$ from stop *A* via stop *E* to stop *C* and vice versa in this network , line $L_2$ from stop *E* to stop *B* and back and line $L_3$ from stop *A* to stop *D* and back. We use following max-algebraic concept to synchronize some directions between lines .

## 2. Max-algebra

We will use necessary notation and formulations from [3,5]. An extensive discussion of the max-algebra can be found in [1,2 ].

Let $\varepsilon = -\infty$ and denote by $\Re_\varepsilon$ the set $\Re \cup \{\varepsilon\}$. For elements $a,b \in \Re_\varepsilon$ we define operations $\oplus$ and $\otimes$ by

$$\begin{aligned} a \oplus b &= \max(a, b), \\ a \otimes b &= a + b. \end{aligned}$$

The structure together with the operation $\oplus$ and $\otimes$ we will called the *max-algebra*. Note that $\varepsilon$ is neutral element for the operation $\oplus$ and absorbing element for $\otimes$ and *0* is neutral element for $\otimes$.

It is possible to extend the max-algebra operations to matrices in the following way. If $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = (b_{ij})$, $\boldsymbol{C} = (c_{ij})$ are matrices of compatible size with entries from $\Re_\varepsilon$ then for all *i*, *j*

$$\mathbf{C} = \mathbf{A} \quad \mathbf{C} = \mathbf{A} \oplus \mathbf{B} \quad \text{if } c_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij}),$$

$$\mathbf{C} = \mathbf{A} \quad \mathbf{C} = \mathbf{A} \otimes \mathbf{B} \quad \text{if } c_{ij} = \sum\nolimits_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj}),$$

where $\sum^{\oplus}$ denotes repeated use of operation $\oplus$. If $\alpha \in \Re_\varepsilon$ then $\alpha \otimes \boldsymbol{A} = (\alpha \otimes a_{ij})$.

- *One-sided linear systems.* Given $\mathbf{A} = (a_{ij}) \in \Re_\varepsilon^{n\times n}$ and $\mathbf{b} \in \Re_\varepsilon^n$, find all $\mathbf{x} \in \Re_\varepsilon^n$ satisfying

$$\mathbf{A} \otimes \mathbf{x} = \mathbf{b}. \tag{1}$$

- *The eigenproblem.* Given $\mathbf{A} = (a_{ij}) \in \Re_\varepsilon^{n\times n}$, find all $\lambda \in \Re_\varepsilon$ (eigenvalues) and all $\mathbf{x} \in \Re_\varepsilon^n$, $\mathbf{x} \neq (\varepsilon, \varepsilon, \ldots, \varepsilon)^T$ (eigenvectors) satisfying

$$\mathbf{A} \otimes \mathbf{x} = \lambda \otimes \mathbf{x}. \tag{2}$$

Let $n \geq 1$ be given integer. The following problems play central role in max-algebra:

- *One-sided linear systems.* Given $\mathbf{A} = (a_{ij}) \in \Re_\varepsilon^{n\times n}$ and $\mathbf{b} \in \Re_\varepsilon^n$, find all $\mathbf{x} \in \Re_\varepsilon^n$ satisfying

$$\mathbf{A} \otimes \mathbf{x} = \mathbf{b}. \tag{1}$$

- *The eigenproblem.* Given $\mathbf{A} = (a_{ij}) \in \Re_\varepsilon^{n\times n}$, find all $\lambda \in \Re_\varepsilon$ (eigenvalues) and all $\mathbf{x} \in \Re_\varepsilon^n$, $\mathbf{x} \neq (\varepsilon, \varepsilon, \ldots, \varepsilon)^T$ (eigenvectors) satisfying

$$\mathbf{A} \otimes \mathbf{x} = \lambda \otimes \mathbf{x}. \tag{2}$$

We formulate our bus models as a solutions of this problems in the next section.

## 3. Modeling of timetables

We will compose two types of timetables on our network with given directions of bus lines. The model BSP (Basic Sinchronization Problem) synchronize departure times on crossing stops of netwok. We use following notation:

- $n$ – the number of different directions in the network, in Figure 2 is $n = 5$,
- $i$ – the number of direction, $i \in \{1,2,\ldots,n\}$,
- $k$ – the order of bus, $k \in \{1,2,\ldots\}$,
- $a_i(k)$ – the traveling time for $k$-th bus in direction $i$,

- $x_i$ (*k*) – the (synchronized) departure time for *k*-th bus in direction *i*..

We will assume possible requirements for changing line by passengers on the example network (in Figure 2) at two stops

*A*: Passengers from terminal stop *E* want connection between stop *A* and destination *D* and vice versa.

*E*: Passengers from terminal stops *A* and *C* want connection between stop *E* and *B* and vice versa.

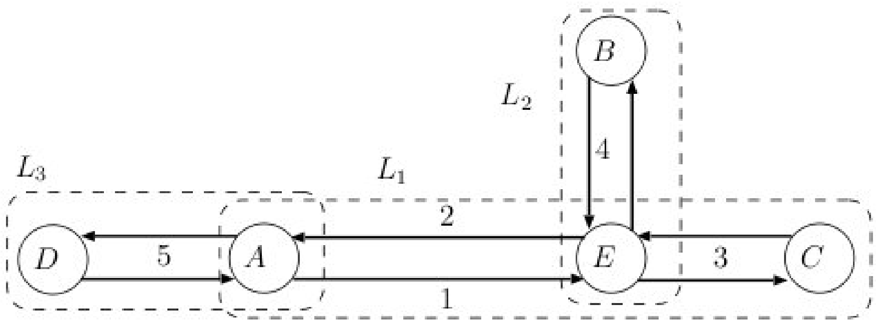


Figure 2: Numbers of different directions in a network

Note the interpretation of directions in Figure 2. These directions 1,5 have starting stop *A* and direction 2,3,4 have starting stop *E*. The traveling times for directions 3,4,5 include waiting on the end stop *C*,*B*,*D* of lines $L_1$, $L_2$, $L_3$.

Now we can formulate the basic sinchronization problem as solution of following linear system (BSP):

$$x_1(k+1) = a_2(k) \otimes x_2(k) \oplus a_5(k) \otimes x_5(k), \tag{3}$$
$$x_2(k+1) = a_3(k) \otimes x_3(k) \oplus a_4(k) \otimes x_4(k), \tag{4}$$
$$x_3(k+1) = a_1(k) \otimes x_1(k) \oplus a_4(k) \otimes x_4(k), \tag{5}$$
$$x_4(k+1) = a_1(k) \otimes x_1(k) \oplus a_3(k) \otimes x_3(k) \oplus a_4(k) \otimes x_4(k), \tag{6}$$
$$x_5(k+1) = a_2(k) \otimes x_2(k) \oplus a_5(k) \otimes x_5(k). \tag{7}$$

The solution of the system estimates the departure times of the (*k*+1)-st buses of all directions as solution of $\boldsymbol{x}(k+1) = \boldsymbol{A}(k) \otimes \boldsymbol{x}(k)$ in matrix notation. Now we will assume that traveling times $a_i(k)$ (i=1,2,...,5) are deterministic and time-independent. Then the behavior of the system BSP is determined by the (max-algebraic) eigenvalue of the matrix **A**. Following section shows how to enumerate this value λ.

## 4. Maximum cycle mean

Given $\boldsymbol{A} = (a_{ij}) \in \Re_\varepsilon^{n\times n}$, let $D_{\boldsymbol{A}} = (N, E. w)$ denotes the weighted digraph with node set $N=\{1,2,\dots,n\}$, arc set $E =\{(i,j)\in N\times N: a_{ij} > \varepsilon\}$and weight $w: :E\to \Re$ such that $w(i,j)= a_{ij}$ for all $(i,j) \in E$.

Let $\pi = (i_1, i_2,..., i_p, i_1)$ denote a cycle in $D_{\boldsymbol{A}}$ of length $l(\pi) = p$. Then weight of the cycle π is $w(\pi,\boldsymbol{A})= a_{i1,i2} + a_{i2,i3} +...+ a_{ip,i1}$ . Let $D_{\boldsymbol{A}}$ has at least one cycle. Let symbol λ(**A**) stand for the *maximum cycle mean* of matrix **A**

$$\lambda(\mathbf{A}) = \max_{\pi} \frac{w(\pi, \mathbf{A})}{l(\pi)}, \tag{8}$$

where the maximization is taken over all cycles in $D_{\boldsymbol{A}}$.

**Theorem [1,2].** *If given* $\boldsymbol{A} = (a_{ij}) \in \Re_\varepsilon^{n\times n}$ *is irreducible or equivalently if* $D_{\boldsymbol{A}}$ *is strongly connected, then there exists one and only one eigenvalue (but possible several eigenvectors) and this eigenvalue is equal to the maximum cycle mean of the digraph* $D_{\boldsymbol{A}}$ .

The value λ (**A**) remain unchanged if the maximization is taken over all elementary cycles. Various algorithm for finding λ (**A**) exist. One of the first and simplest is Floyd-Warshall algorithm [2] based on Vorobyov's formula

$$\lambda(\mathbf{A}) = \max_{k \in N} \max_{i \in N} \frac{a_{ii}^{[k]}}{k}, \tag{9}$$

where $\boldsymbol{A}^k = (a_{ij}^{[k]}) = \boldsymbol{A} \otimes \boldsymbol{A} \otimes \ldots \otimes \boldsymbol{A}$ for $k \in N$. Note that faster maximum mean cycle algorithm than the best known Karp's algorithm is analysed in [4]. Now we can use formula (9) for finding a period of timetable on our simple bus network.

## 5. Example

Let deterministic and the time-independent traveling times of directions (in minutes) begiven $(a_1, a_2, a_3, a_4, a_5) = (20,21,16,18,19)$ in Figure 3. Then the system BSP becomes $\mathbf{x}(k+1) = \mathbf{A} \otimes \mathbf{x}(k)$ with matrix

$$\mathbf{A} = \begin{pmatrix} \varepsilon & 21 & \varepsilon & \varepsilon & 19 \\ \varepsilon & \varepsilon & 16 & 18 & \varepsilon \\ 20 & \varepsilon & \varepsilon & 18 & \varepsilon \\ 20 & \varepsilon & 16 & 18 & \varepsilon \\ 20 & \varepsilon & \varepsilon & \varepsilon & 19 \end{pmatrix}.$$

For this matrix we have that if bus from direction $j$ does not connect route of bus from direction $i$. The corresponding digraf $D_{\boldsymbol{A}}$ is displayed in Figure 4.

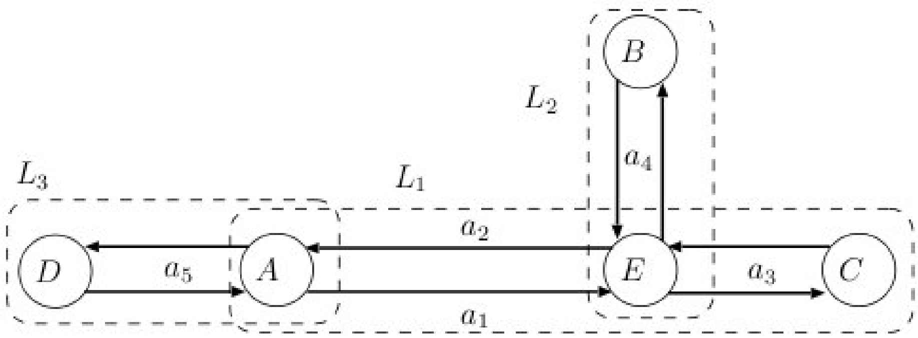


Figure 3: Traveling times for directions of lines

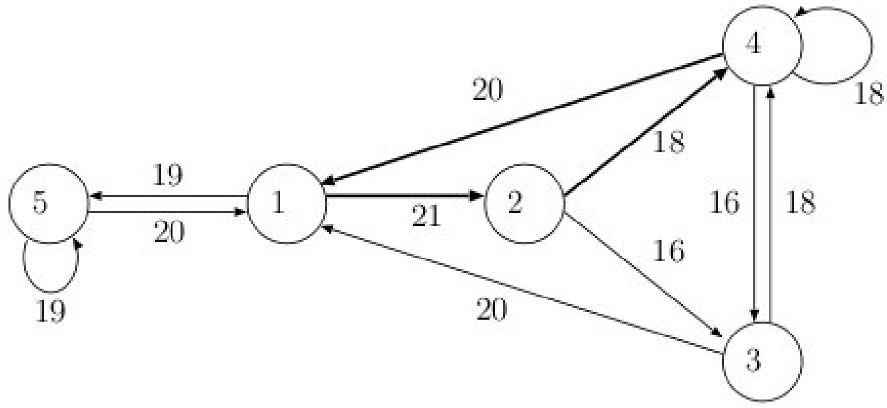


Figure 4: Maximum cycle mean (1,2,4,1) in digraph $D_{\mathbf{A}}$

Eigenvalue of matrix $\lambda(\mathbf{A}) = 59/3 = 19.67$ can be then computed from powers of matrix **A** and formula (9)

$$\mathbf{A}^2 = \begin{pmatrix} 39 & \varepsilon & 37 & 39 & 38 \\ 38 & \varepsilon & 34 & 36 & \varepsilon \\ 38 & 41 & 34 & 36 & 39 \\ 38 & 41 & 34 & 36 & 39 \\ 39 & 41 & \varepsilon & \varepsilon & 39 \end{pmatrix} \quad \mathbf{A}^3 = \begin{pmatrix} 59 & 60 & 55 & 57 & 58 \\ 56 & 59 & 52 & 54 & 57 \\ 59 & 59 & 57 & 59 & 58 \\ 59 & 59 & 57 & 59 & 58 \\ 59 & 60 & 57 & 59 & 58 \end{pmatrix},$$

$$\mathbf{A}^4 = \begin{pmatrix} 78 & 80 & 76 & 78 & 78 \\ 77 & 77 & 75 & 77 & 76 \\ 79 & 80 & 75 & 77 & 78 \\ 79 & 80 & 75 & 77 & 78 \\ 79 & 80 & 76 & 78 & 78 \end{pmatrix} \quad \mathbf{A}^5 = \begin{pmatrix} 98 & 99 & 96 & 98 & 97 \\ 97 & 98 & 93 & 95 & 96 \\ 98 & 100 & 96 & 98 & 98 \\ 98 & 100 & 96 & 98 & 98 \\ 98 & 100 & 96 & 98 & 98 \end{pmatrix}.$$

This means that for an appropriate choice of $\mathbf{x}(1)$ every 19.67 minutes the bus can depart.

## 6. Open problem

We described ideas of modeling a bus transportation network with max-linear model how to apply corresponding max-algebraic eigenvalues and eigenvectors to enable passengers to change bus lines for making timetables on given lines.

Now we will discuss the timetable information. Let **d**(1) denote the vector which contains the scheduled departure time of the first buses in each direction. If we want our schedule to be as regular as possible the timetable shoud satisfy

$$\mathbf{d}(k+1)=\tau \otimes \mathbf{d}(k), \quad k=1,2,3,\ldots \qquad (10)$$

where $\mathbf{d}(k)$ is a vector of scheduled departure times $k$-th buses in all directions and $\tau$ represents the period of timetable that should satisfy $\tau \geq \lambda(\mathbf{A})$. Then the modified system BSP becomes

$$\mathbf{x}(k+1)=\mathbf{A}\otimes\mathbf{x}(k) \otimes\mathbf{d}(k+1). \quad k=1,2,3,\ldots \qquad (11)$$

After the choice of the period, we have to choose the initial value **d**(1) such that when $\mathbf{x}(1)=\mathbf{d}(1)$ and there are no delays

$$\mathbf{x}(2) = \mathbf{A}\otimes\mathbf{x}(1) \otimes\mathbf{d}(2) = \mathbf{A}\otimes\mathbf{d}(1) \otimes\mathbf{d}(2) = \mathbf{d}(2)$$

or in other words

$$\mathbf{A}\otimes\mathbf{d}(1) \leq \mathbf{d}(2). \qquad (12)$$

If condition (12) is satisfied then the timetable is feasible. Then from (10) can be concluded that $\mathbf{A}\otimes\mathbf{d}(k) \leq \mathbf{d}(k+1)$ for all $k\geq 1$. When (10) does not hold, we have to require (10).

Combination (10) and (12) define a set of feasible timetables. Open question is how to enumerate this in an effective way.

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# MULTICRITERIA APPROACHES TO COMPETITIVENESS

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**Abstract:** Regional competitiveness is the source of national competitiveness. This paper presents multi-criteria decision making methods for evaluation of the regional competitiveness. Specific coefficients reflect economic productivity of the region in form of factors of production inside of the region. The technology for evaluation of regional competitiveness is based on application of three methods of multi-criteria decision making. The first one is the classical weighted average where relevance of criteria's significance is determined by the method of Ivanovic deviation. The second method - FVK is a multiplicative version of AHP, the third method is the well known DEA. The results of the methods are compared with the simple averaging. On the basis of the multi-criteria techniques, we obtain a more detailed perspective in the study of the competitiveness of the NUTS3 regions in the Czech Republic and also NUTS2 regions (V4 - Visegrad Four countries) in EU within the time period of 7 years (2000 – 2006).

**Keywords:** competitiveness, multi-criteria methods, Ivanovic deviation, DEA.

## 1. Introduction - competitiveness of regions.

This paper deals with multi-criteria decision making methods for evaluating the regional competitiveness. Specific coefficients reflect economic productivity of the region in form of factors of production inside the region (effect of one-regional unit) and are revitalized by the capacity of real employment in the region. Particularly, we deal with the coefficient of effective disposability, effectiveness of economical development, effectiveness of investments, effectiveness of revenues and effectiveness of construction works. The technology of evaluation of regional competitiveness is based on application of three methods of multi-criteria decision making. The first one is the classical weighted average method where the relevance of criteria's significance is determined by the method of Ivanovic deviation. The second method - FVK is a multiplicative version of AHP. The third method is the well known Data Envelopment Analysis - DEA. The results of all three methods have been compared with the simple averaging method.

## 2. Specific indicators for evaluation of regional competitiveness.

There does not exist a "universal" methodology for assessing degree of regional non-competitiveness. Some "alternative way" for evaluating regional competitiveness is to define a group of specific economic indicators of efficiency. The basic idea is to assess the internal sources of regional competitiveness in detail, see [3]. The evaluation of the competitiveness through 5 specific relative indicators KER, KED, KIV, KET and KSP have been proposed and discussed in [6].

Another technique of evaluation of regional competitiveness is Ivanovic deviation (ID), see [6]. This method belongs to the techniques of multi-criteria decision-making and its purpose here is to assess the ranks of the regions, too. In comparison with the simple averaging, it takes into account importance and mutual dependence of the decision-making criteria (i.e. 5

specific coefficients already mentioned). First, the criteria (i.e. specific coefficients) should be ranked according to their relative importance. This ranking is done by an expert evaluation. Here, KER is the most important coefficient as it reflects total economic efficiency of the region and it also includes the level of production. The second most important criterion is KED, the disposable income - another source for the household consumption and transfer savings to investment. KIV is the gross fixed capital - an indicator of connection of expenditures for creation of the fixed assets. These assets are also included in the regional production. KET could be interpreted as the result of realized production. KSP is the criterion of employment. In this method, the weight of each criterion based on its relative importance - ranking takes into account correlation coefficients with the previous (i.e. more important) criteria. Then the weighted distance of the current variant to the ideal (fictitious) one is calculated as follows, see [6]:

$$I_j = \frac{\left|x_1^f - x_{1j}\right|}{s_1} + \sum_{i=2}^{n} \frac{\left|x_i^f - x_{ij}\right|}{s_i} \prod_{k=1}^{i-1} (1 - |r_{ki}|), \tag{1.1}$$

$x_i^f$ – value of i-th criterion of ideal (fictitious) variant (i.e. region),
$x_{ij}$ – value of i-th criterion j-th variant,
$r_{ki}$ – correlation coefficient i-th a k-th criterion (i.e. specific coefficient),
$s_i$ – standard deviation i-th criterion calculates:

$$s_i = \sqrt{\frac{1}{m} \sum_{j=1}^{m} \left(x_i^j\right)^2 - \left(\bar{x}_i\right)^2} \tag{1.2}$$

where $m$ – total value of variants, $n$ – total number of criteria.

The approach based on the application of Ivanovic deviation seems to be more relevant comparing to the results of the method of simple averaging. As we know the importance of the criteria and correlations (i.e. dependences) among the criteria we are able to determine the “distance” to the ideal region in a more realistic way. Then the final rank of regions corresponds to the different economic importance of individual criteria (i.e. specific coefficient of efficiency). Thanks to this fact we consider the final rank as another contribution of this alternative approach to evaluation of regional competitiveness of the NUTS3 regions in the Czech republic (PHA, STČ, ULK, ..., KVK), see Table 1.

### 3. AHP and FVK

In this section we deal with the same problem applying another alternative method. The Analytic hierarchy process (AHP method) was published already in 1980s, see [10], recently, it is considered as the "classical" decision making methodology. On the other hand, the FVK is a newly created tool extending application possibilities of the classical AHP, see [9]. Here, we compare and discuss the results obtained by this method with the previous Ivanovic deviation method (ID).

Comparing ID and FVK methods there are some significant differences:

- In classical AHP the weights $w_k$ are calculated from the pair-wise comparison matrix $S = \{s_{ij}\}$ by the principal eigenvector method, see [9], whereas in FVK the weights $w_k$ are calculated from the pair-wise comparison matrix **S** through the geometric mean as:

$$w_k = \frac{\left(\prod_{j=1}^{n} s_{kj}\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} s_{ij}\right)^{1/n}} \tag{3.1}$$

where $k = 1,2,\ldots,m$, see [8]. The elements of the pair-wise comparison matrix $S$ are evaluated by expert pair-wise comparisons of the relative importance of the criteria – i.e. the specific coefficients.

- The total evaluation $J_i$ of every variant (i.e. 14 regions in the CR, $i$ = 1,2,...,14) is calculated as the weighted average:

$$J_i = \sum_{k=1}^{n} w_k a_{ik} \tag{3.2}$$

where $a_{ik}$ is the normalized value of the $k$-th specific coefficient for the $i$-th region described in Section 2.

- FVK method reduces some theoretical disadvantages of the method of principal eigenvector used in AHP, e.g. the rank reversal problem.

All results presented in Table 1, column 6 and 7, have been calculated by software tool named FVK. This SW has been created as an add-in of MS Excel 2003 within the GACR project No. 402060431, see [8].

| 1. rank /method | 2. average 5 coeff. 2000 | 3. average 5 coeff. 2006 | 4. Ivanovic deviation 2000 | 5. Ivanovic deviation 2006 | 6. FVK 2000 | 7. FVK 2006 |
|---|---|---|---|---|---|---|
| 1. | PHA | PHA | PHA | PHA | PHA | PHA |
| 2. | STČ | JHM | STČ | STČ | STČ | JHM |
| 3. | ULK | STČ | ULK | MSK | ULK | STČ |
| 4. | JHČ | MSK | MSK | PAK | JHČ | MSK |
| 5. | PLK | PLK | PLK | ULK | JHM | PLK |
| 6. | MSK | ULK | LBK | PLK | PLK | ULK |
| 7. | JHM | PAK | HKK | LBK | MSK | PAK |
| 8. | LBK | LBK | JHČ | VYS | ZLK | LBK |
| 9. | ZLK | ZLK | PAK | ZLK | PAK | ZLK |
| 10. | PAK | VYS | ZLK | JHM | LBK | JHČ |
| 11. | HKK | JHČ | JHM | HKK | HKK | VYS |
| 12. | VYS | OLK | VYS | JHČ | OLK | OLK |
| 13. | OLK | HKK | OLK | OLK | VYS | HKK |
| 14. | KVK | KVK | KVK | KVK | KVK | KVK |

**Table 1**. Final ranks of regions using selected methods

## 4. DEA

Data Envelopment Analysis (DEA) is a relatively new "data oriented" approach for evaluating the performance of a set of peer entities called Decision Making Units (DMUs) converting multiple inputs into multiple outputs. Here, we applied DEA to all 35 central European NUTS2 regions in Visegrad Four countries (V4). Recent years have seen a great variety of applications of DEA for use in evaluating the performances of many different kinds of entities engaged in many different activities in many different contexts in many different countries, see [2]. These DEA applications have used DMUs of various forms to evaluate the performance of entities, such as hospitals, US Air Force wings, universities, cities, courts, business firms, and others, including the performance of countries, regions, etc.

As pointed out in [2], DEA has also been used to supply new insights into activities (and entities) that have previously been evaluated by other methods. Since DEA in its present form was first introduced in 1978, see [1], researchers in a number of fields have quickly recognized that it is an excellent and easily used methodology for modeling operational processes for performance evaluations. In their originating study [1], DEA is described as a "mathematical programming model applied to observational data that provides a new way of obtaining empirical estimates of relations - such as the production functions and/or efficient production possibility surfaces – that are cornerstones of modern economics".

Relative efficiency in DEA accords with the following definition, which has the advantage of avoiding the need for assigning a priori measures of relative importance to any input or output.

*Definition 1.* (Efficiency – Extended Pareto Koopmans Definition): Full (100%) efficiency is attained by any DMU if and only if none of its inputs or outputs can be improved without worsening some of its other inputs or outputs.

In most management or social science applications the theoretically possible levels of efficiency is not known. The preceding definition is therefore replaced by emphasizing its uses with only the information that is empirically available as in the following definition:

*Definition 2.* (Relative Efficiency): A DMU is to be rated as fully (100%) efficient on the basis of available evidence if and only if the performances of other DMUs does not show that some of its inputs or outputs can be improved without worsening some of its other inputs or outputs.

Our model is based on the inputs and outputs, they must be chosen carefully with regard to their definition in economic theory. This fact is vital for us to perceive the efficiency like a "mirror" of competitiveness. Moreover, here we present only one version of DEA model, particularly, the most popular input oriented CCR model and also output oriented CCR model, see [1]. For more detailed analysis other DEA models are appropriate, for revealing differences in efficient units the super-efficiency models can be applied, see e.g. [2].

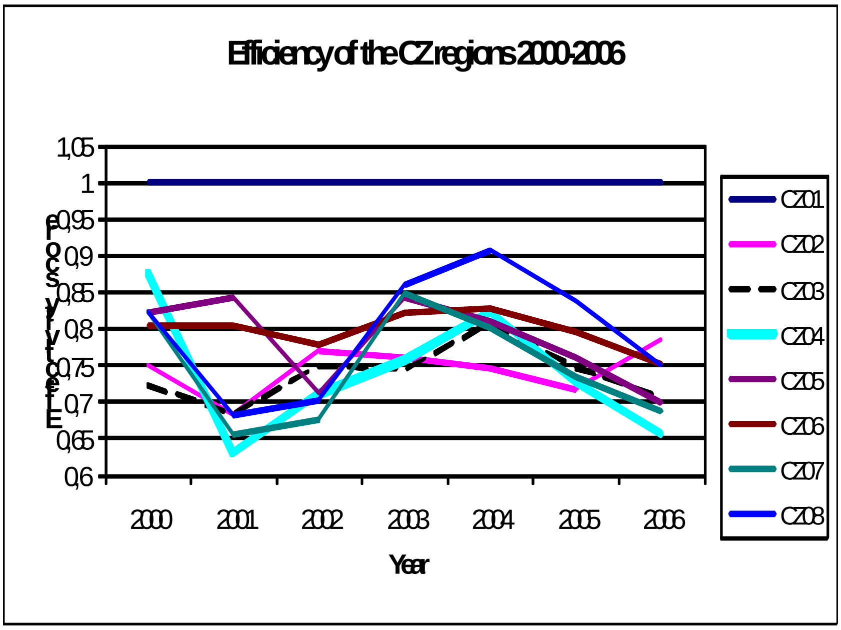
Now we introduce criteria for selecting inputs and outputs used in DEA model applied to efficiency of NUTS2 regions in V4 (i.e. Czechia, Slovakia, Poland, Hungary). It is evident that the overall performance of the regional economy affects the number of people employed in various sectors, their skills and working age (15-55 years). Therefore, we selected the criterion of employment rate and criterion of the creation of the GFCF (Gross Fixed Capital Formation). The GFCF includes generally investment activity of domestic companies and fixed assets of foreign companies, in addition to the GFCF is the "engine" of the innovation competitiveness. The GFCF is largely influenced by the inflow of foreign investment, especially foreign direct investment. Efficiency will demonstrate the ability of the NUTS2 region to transform its own or profitable capital for its further development. The third included input is the net disposable income of households. In terms of competitiveness the disposable income plays an important role, especially because it directly reflects the purchasing power of the region.

There are two outputs in our DEA model. Reflected outputs are measured by GDP in purchasing parity standards and labor productivity per person employed. The GDP is the most important macroeconomic aggregate, and if it is measured per region, we can take into account the limited number of inputs, due to which it was achieved. Similarly, the labor productivity may be dealt with, as it shows us how much production economically active people have created, or employed persons in the national economy, respectively. In terms of regions we also take into account which national economy sectors were involved the most or least in the production.

In Figure 1 we compare the Czech and Slovak region efficiency and in Table 2 it is evident that the best results are traditionally achieved by economically powerful "capital" regions being efficient during the whole period 2000 – 2006.

## 5. Conclusion

The paper is aimed to present multi-criteria approaches to evaluation of competitiveness (efficiency) of European regions (NUTS2, NUTS3). This evaluation was based on the applications of 3 models (Ivanovic deviation, FVK, DEA) calculating an efficiency index of each region. In the absence of the mainstream in a methodological approach to regional competitiveness, this paper should be understood as a contribution to discussion about quantitative measurement of competitiveness at the regional level.



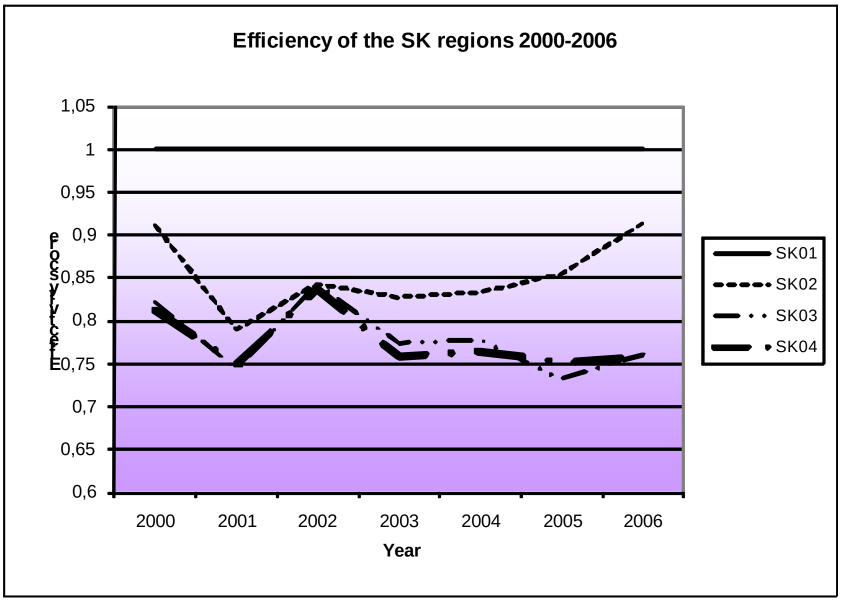




**Figure 1**. Efficiency of Czech and Slovak regions 2000-2006 (CCR, Input oriented model)

| Region rank | Code of NUTS 2 | Name of region | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | CZ01 | Praha | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 |
| 2 | CZ02 | Střední Čechy | 0,749 | 0,681 | 0,768 | 0,761 | 0,747 | 0,716 | 0,784 |
| 3 | CZ03 | Jihozápad | 0,723 | 0,682 | 0,750 | 0,743 | 0,808 | 0,747 | 0,705 |
| 4 | CZ04 | Severozápad | 0,875 | 0,628 | 0,712 | 0,758 | 0,822 | 0,725 | 0,656 |
| 5 | CZ05 | Severovýchod | 0,822 | 0,844 | 0,710 | 0,841 | 0,811 | 0,759 | 0,700 |
| 6 | CZ06 | Jihovýchod | 0,804 | 0,804 | 0,780 | 0,822 | 0,829 | 0,796 | 0,752 |
| 7 | CZ07 | Střední Morava | 0,821 | 0,655 | 0,675 | 0,848 | 0,803 | 0,734 | 0,687 |
| 8 | CZ08 | Moravskoslezsko | 0,822 | 0,682 | 0,703 | 0,861 | 0,906 | 0,838 | 0,750 |
| 9 | HU10 | Közép-Magyarország | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 |
| 10 | HU21 | Közép-Dunántúl | 0,873 | 0,860 | 0,770 | 0,747 | 0,774 | 0,711 | 0,682 |
| 11 | HU22 | Nyugat-Dunántúl | 0,886 | 0,808 | 0,817 | 0,881 | 0,829 | 0,725 | 0,699 |
| 12 | HU23 | Dél-Dunántúl | 1,000 | 0,991 | 0,939 | 0,823 | 0,792 | 0,704 | 0,657 |
| 13 | HU31 | Észak-Magyarország | 0,846 | 0,878 | 0,842 | 0,792 | 0,792 | 0,772 | 0,679 |
| 14 | HU32 | Észak-Alföld | 0,905 | 0,837 | 0,822 | 0,783 | 0,785 | 0,707 | 0,647 |
| 15 | HU33 | Dél-Alföld | 1,000 | 0,994 | 0,883 | 0,830 | 0,828 | 0,769 | 0,756 |
| 16 | PL11 | Lódzkie | 0,895 | 0,837 | 0,974 | 0,968 | 0,908 | 0,969 | 1,000 |
| 17 | PL12 | Mazowieckie | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 |
| 18 | PL21 | Malopolskie | 0,776 | 0,816 | 0,823 | 0,829 | 0,878 | 0,879 | 0,887 |
| 19 | PL22 | Sląskie | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 |
| 20 | PL31 | Lubelskie | 0,913 | 0,956 | 1,000 | 0,995 | 0,917 | 0,989 | 0,992 |
| 21 | PL32 | Podkarpackie | 0,901 | 0,938 | 0,881 | 0,859 | 0,863 | 0,863 | 0,855 |
| 22 | PL33 | Swietokrzyskie | 0,835 | 0,996 | 0,820 | 0,956 | 0,852 | 0,849 | 0,871 |
| 23 | PL34 | Podlaskie | 0,907 | 1,000 | 0,939 | 0,896 | 0,821 | 0,859 | 0,902 |
| 24 | PL41 | Wielkopolskie | 0,768 | 0,732 | 0,830 | 0,815 | 0,909 | 0,889 | 0,804 |
| 25 | PL42 | Zachodniopomorskie | 0,996 | 1,000 | 1,000 | 1,000 | 0,923 | 0,978 | 1,000 |
| 26 | PL43 | Lubuskie | 1,000 | 1,000 | 0,990 | 0,917 | 0,910 | 0,918 | 0,899 |
| 27 | PL51 | Dolnoslaskie | 0,872 | 0,942 | 0,909 | 0,944 | 0,903 | 0,908 | 0,910 |
| 28 | PL52 | Opolskie | 0,899 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 |
| 29 | PL61 | Kujawsko-Pomorskie | 0,937 | 0,938 | 0,957 | 1,000 | 1,000 | 0,991 | 0,951 |
| 30 | PL62 | Warminsko-Mazurskie | 1,000 | 1,000 | 0,941 | 0,944 | 0,893 | 0,923 | 0,914 |
| 31 | PL63 | Pomorskie | 0,920 | 0,894 | 0,889 | 0,978 | 0,944 | 0,944 | 0,924 |
| 32 | SK01 | Bratislavský kraj | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 |
| 33 | SK02 | Západní Slovensko | 0,912 | 0,788 | 0,843 | 0,826 | 0,833 | 0,854 | 0,912 |
| 34 | SK03 | Střední Slovensko | 0,821 | 0,744 | 0,843 | 0,774 | 0,778 | 0,732 | 0,760 |
| 35 | SK04 | Východní Slovensko | 0,812 | 0,749 | 0,836 | 0,759 | 0,764 | 0,751 | 0,758 |

**Table 2.** Application of DEA for NUTS 2 regions (CCR, Input oriented model)

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# A MULTI-LEVEL MULTI-PRODUCT APPROACH TO SITTING COLLECTION POINTS IN REVERSE LOGISTICS SYSTEMS

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**Abstract:** This paper presents an approach to establishing reverse logistics system through defining optimal locations of collection points, transfer points and recycling centers. In order to model the influence of distance between consumers and collection points on three types of facilities to be located in this system, we introduce the collection point's catchment area. A multi level location model was proposed and tested on the example of New Belgrade municipality.

**Keywords:** reverse logistics, end of life (EOL) products, facility location

## 1. Introduction

Due the changes in the production and consumption patterns nowadays, which resulted in shortening the products life cycle and earlier products discarding, increasing environmental concerns about the disposal of large quantities of products have resulted in efforts to take back end-of-life (EOL) consumer products. Legislation aimed at forcing manufacturers to take back EOL products has been implemented in many countries. This, along with decreasing landfill space, underscores the importance of developing efficient methods and models for the management of EOL products. The solution to this problem offers reverse logistics, because products recovery, in any form, benefits both from the environmental as well as socio-economic perspective. The main goals of reverse logistics system are to reduce the total distance of transportation, increase the quantities of EOL products collected, reduce the amount of EOL products carried to treatment facilities inefficiently and to connect reverse logistics to forward logistics in an efficient way [1].

From there, the main intention of this paper was to analyze modelling approach that could be used to establish three level reverse logistics network for EOL products, composed of a set of collection points, sorting points and recycling facilities. In this way, research presented in this paper should be understood as an extension of the previous effort in this field [2]. There are two main directions in which those extensions are made. The first is in strict consideration of multiproduct system nature by introducing two index location variables, and second is model application based on numerical example.

Remaining part of the paper was organized in following way. Section two describes structure of reverse logistic networks for EOL products, and introduces some requirements which should be satisfied. Section three introduces modelling approach, while numerical results of modelling approaches proposed are shown in section four. Ending part of the paper gives some concluding remarks.

## 2. EOL reverse logistics network

Facility location is an important issue in the reverse logistics networks design, because appropriate facility location can save costs, improves treatment efficiency and improves customers satisfaction. A number of models for reverse logistics networks design are proposed in the literature and most of them formulate discrete facility location-allocation in order to obtain the optimal infrastructure design ([3], [4], [5]). Also, most of them are case based which makes it difficult to apply them to other situations or cases and, some of them, but not all incorporate collection facilities in their formulations. Operations research techniques are used to determine the optimum number and location of facilities. In practice, only a limited number of sites will be feasible, due to a number of factors such as access, topography, cost, and environmental acceptability. However, sitting of any collection, transfer or treatment facility cannot depend solely on technical and economic analyses, but must include public participation in the selection process. In most researches, the active participation of products end users is always assumed, which is not always the case in reality. Personal inconveniences associated with required time, storage space, access to collection points are some of the factors that have influence on participation in collection programs. In paper [6] authors found that distance to bring sites is the main obstacle to effective recycling. Also, in [7] authors found that approximately 94% of interviewed individuals would walk a maximum of five minutes (10 minutes for a round trip) and 55 % took more product to bring sites to be recycled if those sites are close to their residence. So, in order to model the influence of distance between users and collection points on the optimal locations of facilities to be located, we introduce the collection point's catchment area (Fig.1). The catchment area models the influence of distance between end users and collection points, in the sense that for all end users and collection points, collection service may exist only when end users are within the certain (reasonable) distance from a collection point *k*. Therefore, catchment area denotes the area within the circle of certain predefined radius from the drop-off location. That is, any arbitrary end user can be allocated to the collection point only if it is located within the collection point’s catchment area.

$R_k$ – radius of the collection points catchment area *k*
$d_{ki'}$ – distance between collection point *k* and end user *i’*
$d_{ki''}$ – distance between collection point *k* and end user *i”*

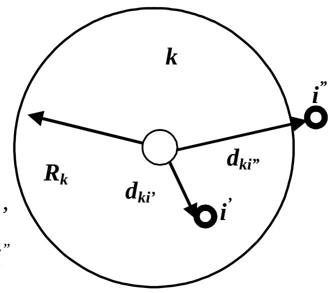


**Fig 1:** Collection point’s catchment area

## 3. Mathematical formulation

Solving problem of locating collecting, transfer and treatment facilities may be realized in different ways by using different problem formulations and modeling approaches. In this paper, we propose following approach (Fig. 2).

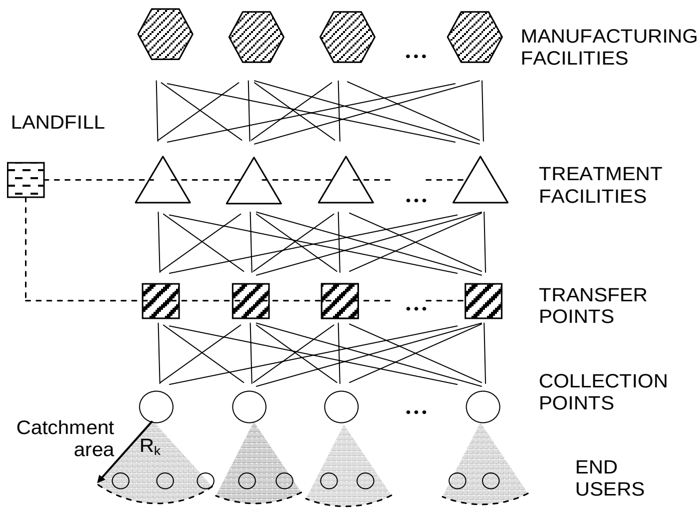


**Fig. 2 Reverse logistics network for EOL products**

The folowing notation is used to describe a model:

Sets:

$I = \{1,...N_i\}$ end users zones

$K = \{1,...N_k\}$ potential collection points

$L = \{1,...N_l\}$ potential transfer points

$J = \{1,...N_{j+1}\}$ potential treatment facilities plus disposal option

$S = \{1,...N_s\}$ manufacturing facilities

$P = \{1,...N_p\}$ product types

Parameters

$p$ product type, $p \in P$

$\alpha_p$ minimal disposal fraction of product type $p$, $p \in P$

$G_{pk}$ capacity of collection point $k$, $k \in K$

$G_{pl}$ capacity of transfer point $l$, $l \in L$

$G_{pj}$ capacity of treatment facility $j$, $j \in J$

$G_{ps}$ capacity of manufacturing facility $s$, $s \in S$

$C_{pkl}$ transportation costs of transporting EOL product $p$ from collection point $k$ to transfer facility $l$, $p \in P$, $k \in K$, $l \in L$

$C_{plj}$ transportation costs of transporting EOL product $p$ from transfer point $l$ to treatment facility $j$, $p \in P$, $l \in L$, $j \in J$

$C_{plj+1}$ transportation costs of transporting EOL product $p$ from transfer point $l$ to landfill site $j+1$, $p \in P$, $l \in L$, $j+1 \in J$

$C_{pjs}$ transportation costs of transporting EOL product $p$ from treatment facility $j$ to manufacturing facility $s$, $p \in P$, $j \in J$, $s \in S$

$d_{ik}$ distance between end user zone $i$ to collection point $k$, $i \in I$, $k \in K$

$R_k$ radius of the catchment area for collection point $k$, $k \in K$

Variables

$X_{pik}$ fraction of product $p$ transported from end user zone $i$ to collection site $k$. The collection site $X_{piD}$ is a dummy site with infinite cost and infinite capacity, and prevents infeasibility in the solution procedure due to insufficient capacity, $p \in P, i \in I, k \in K$ (unsatisfied demands)

$X_{pkl}$ fraction of product $p$ transported from collection site $k$ to transfer point $l$, $p \in P, k \in K, l \in L$

$X_{plj}$ fraction of product $p$ transported from transfer point $l$ to treatment facility $j$, $p \in P, l \in L, j \in J$

$X_{pjs}$ fraction of product $p$ transported from treatment facility $j$ to manufacturing facility $s$, $p \in P, j \in J, s \in S$

$Y_{pk}$ binary variable, $Y_{pk}$=1 if collection point $k$ is opened, otherwise $Y_{pk}$ =0, $k \in K$

$Y_{pl}$ binary variable, $Y_{pl}$ =1 if transfer point $l$ is opened, otherwise $Y_{pl}$ =0, $l \in L$

$Y_{pj}$ binary variable, $Y_{pj}$ =1 if treatment facility $j$ is opened, otherwise $Y_{pj}$ =0, $j \in J$ Then, the formulation of the problem as a mixed integer linear programming problem is given by

$$\min \sum_k \sum_l \sum_p C_{pkl} X_{pkl} + \sum_j \sum_s \sum_p C_{pjs} X_{pjs} +$$
$$\sum_l \sum_j \sum_p C_{plj} X_{plj} + \sum_l \sum_p C_{plj+1} X_{plj+1} \qquad (1)$$

s.t.

$$\sum_k \mathrm{X_{pik}} + \mathrm{X_{piD}} = \mathrm{q_{ip}} \ldots \forall \mathrm{i, p} \qquad (2)$$

$$\sum_i \mathrm{X_{pik}} - \sum_l \mathrm{X_{pkl}} = 0 \ldots, \forall \mathrm{p}, \forall \mathrm{k} \qquad (3)$$

$$\sum_k X_{pkl} - (1-\alpha_p)\sum_j X_{plj} = 0\ldots,\forall p,\forall l \quad (4)$$

$$\sum_l \mathrm{X_{plj}} - \sum_s \mathrm{X_{pjs}} = 0\ldots,\forall \mathrm{p}, \mathrm{j} \quad (5)$$

$$X_{pik} \le Y_{pk}G_{pk}\ldots,\forall i,k,p \quad (6)$$

$$X_{pkl} \le Y_{pl}G_{pl}\ldots,\forall k,l,p \quad (7)$$

$$X_{plj} \le Y_{pj}G_{pj}\ldots,\forall l,j,p \quad (8)$$

$$\sum_i X_{pik} \le Y_{pk}G_{pk}\ldots\forall k,p \quad (9)$$

$$\sum_k X_{pkl} \le Y_{pl}G_{pl}\ldots\forall l,p \quad (10)$$

$$\sum_l X_{plj} \le Y_{pj}G_{pj}\ldots\forall j,p \quad (11)$$

$$\sum_j X_{pjs} \le G_{ps}\ldots\forall s,p \quad (12)$$

$$(d_{ik} - R\ )X_{pik} \le 0,\ldots\forall i,k,p \quad (13)$$

$$Y_{pk}, Y_{pl}, Y_{pj} \in \{0,1\} \quad (14)$$

$$X_{pik}, X_{pkl}, X_{plj}, X_{plj+1}, X_{pjs} >= 0 \quad (15)$$

Objective function (1) minimizes transportation costs of transporting products from end user zones to manufacturing facilities, via collection points, transfer and treatment facilities. Equitation (2) ensures that all EOL products currently located at end user zones are transferred to collection points. Constraints (3)-(5) are flow conservation constraints, for collection point level, transfer point and treatment facility level respectively. Equitation (4) models minimum disposal fraction form transfer point level. Constraints (6) to (12) are capacity and opening constraints, but since we are not determining locations of manufacturing facilities, no opening constraints are used for this type of facility. And finally, constraint (13) represents the catchment area of collection point *k* and allocation of end users to collection points. Constraint set (15) requires the decision variable *X* to be continuous between zero and one, while constraint set (14) enforce the binary restriction on the *Y* decision variables.

## 4. Numerical results

Proposed formulation was tested on the New Belgrade municipality example, the most populated Belgrade city municipality. We have 80 users' zones, 2 types of products, 2 manufacturing facilities and one landfill site. There is a limit to the number of collection (170), transfer (7) and recycling sites (4) that can be opened, but the choice of which collection sites, which sorting and recycling sites to be opened must be decided by the model. Model has 30498 constraints and 30188 variables, among which 362 were integer and binary. Model (1)-(15) was solved through open source solver GLPK and numerical results are shown in Table 1.

| **Product type** | **P1** | | | | | | |
|---|---|---|---|---|---|---|---|
| **Catchment's area** | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 |
| **Coll. points opened** | 44 | 47 | 43 | 25 | 31 | 31 | 32 |
| **Transf.points opened** | 3 | 6 | 4 | 5 | 6 | 5 | 5 |
| **Treatm. fac. opened** | 1 | 3 | 3 | 3 | 2 | 2 | 3 |
| **Served end users (%)** | 61.3 | 97.3 | 98.8 | 100 | 100 | 100 | 100 |

| **Product type** | **P2** | | | | | | |
|---|---|---|---|---|---|---|---|
| **Catchment's area** | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 |
| **Coll. points opened** | 44 | 48 | 49 | 43 | 34 | 37 | 33 |
| **Transf.points opened** | 4 | 7 | 7 | 7 | 6 | 6 | 6 |
| **Treatm. fac. opened** | 3 | 2 | 3 | 4 | 4 | 3 | 2 |
| **Served end users (%)** | 61.25 | 97.25 | 98.75 | 100 | 100 | 100 | 100 |

**Table 1. Impact of the catchment area radius on reverse logistics network design**

From the Table 1 importance of the catchment area radius becomes obvious, because of its huge impact on collected quantity of recyclables from the one side, and on the logistic network configuration from the other. Also, results shown in the table 1 indicate importance of transport costs accuracy, facilities' capacities and other parameters used in defining in modelling and shaping reverse logistics networks, and in collection process efficiency assessing.

## 5. Conclusion

This paper presents a multi level, multi product facility location model reverse logistics network design. The proposed model aims finding effective strategies for the return of discarded products from end users to recycling factories, via collection points, transfer facilities and treatment facilities, with minimal costs. The main contribution of this paper is in testing the model which respects multiproduct reverse logistics system on real example, and in analyzing impact of collection point's catchments area. Of course, proposed approach should be understood only as beginning of the more thoroughly research which is just opened, where one possible extension may be related to analyze possibilities for defining collection points catchment area as a function of socio demographic and other relevant characteristics of potential users.

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# MODELING OF AVERAGE VALUE OF PENSION UNITS OF GROWTH PENSION FUNDS IN SLOVAKIA

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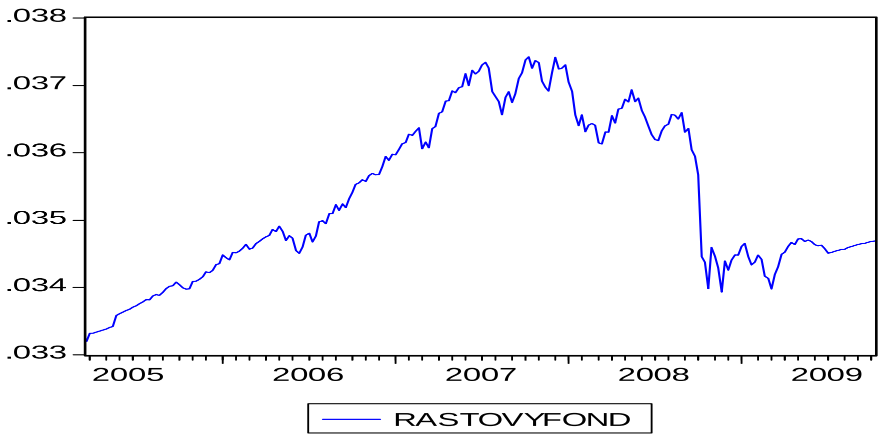
## Abstract

Report treats of modeling the development of time series of growth rates computed from the time series of weekly average values of pension units of growth pension funds in Slovakia during the period from March 22, 2005 till May 15, 2009 with 237 observations. Time series is stationary in the mean but not in the variance, so combination of models AR and ARCH have been used. Conditional mean of the series was modeled by AR(2) model with constant and the conditional variance was modeled by ARCH (3) model. In addition, we have assumed that the growth rates in the average value of pension units of growth pension funds could be influenced by means of average price of crude oil. Because of positive results of Granger causality test about the hypothesis that average world's price of crude oil could influenced the growth rates in the average value of pension units of growth pension funds in Slovakia, we have estimated also model ADL (2) with exogenous variable of average world's price of crude oil. Moreover it was showed, that model without and with exogenous variable are giving very similar results of Theil inequality coefficient.

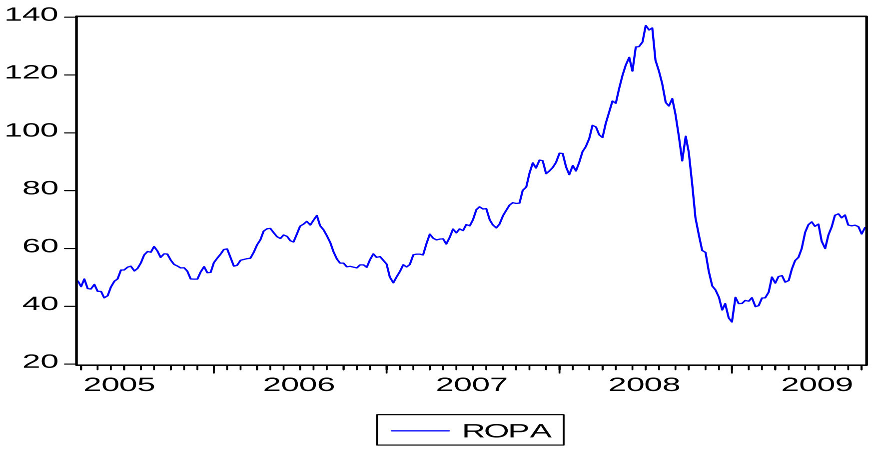
**Keywords:** weekly average value of pension units of growth pension funds, growth rates, models AR (p), ARCH(q), Theil inequality coefficient

## 1. Source of Data

Variables used in this article are taken from the work [2]. There are: variable of weekly average value of pension units from six growth Pension Funds in Slovakia (called rastovyfond) and variable of weekly average world's price of crude oil (called ropa). The development of these two aggregated variables is on Picture 1 and 2.

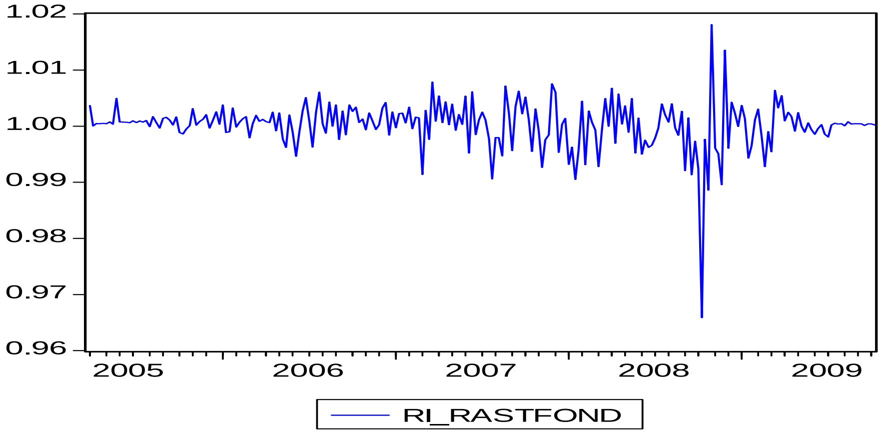



**Picture** Weekly average value of pension units of six Growth Pension Funds in Slovakia

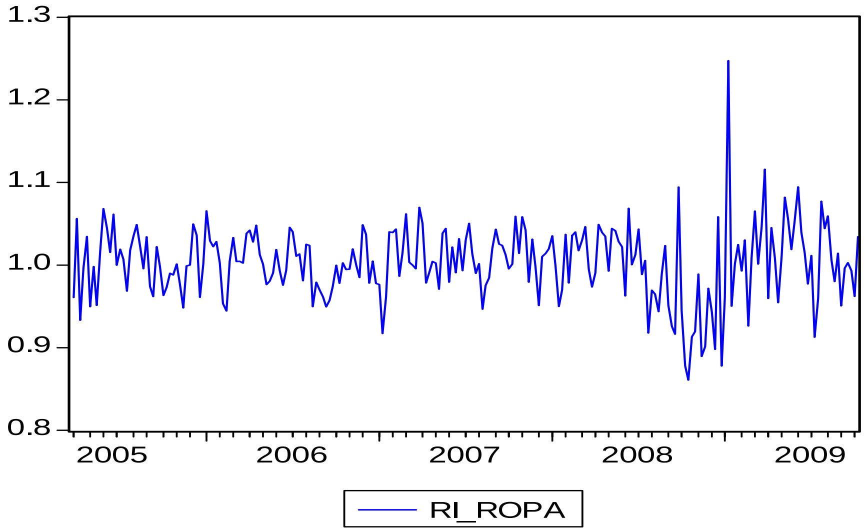



**Picture 2** Weekly average world's price of crude oil

Both variables are nonstationary in the mean as well as in the variance, but growth rates for both variables are nonstationary only in the variance, as is showed on Picture 3 and 4.




**Picture 3** Growth rates of the aggregate variable *rastovyfond*




**Picture 4** Growth rates of the aggregate variable *ropa*

For modeling heteroscedasticity of the variable *ri_ratsfond* on picture 3 we will use model ARCH to get confidence interval of the risk to hold an asset.

## 2. ARCH Model

In developing an ARCH model, we have to consider two distinct equations – one for the conditional mean and one for the conditional variance. Conditional mean of the series without seasonality is usually described by ARIMA(p, d, q) of the form

$$\phi_p(B)\theta_q(B)(1-B)^d y_t = K + a_t \qquad (1)$$

where

$\phi_p(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p$ is autoregressive polynomial of order $p$,

$\theta_q(B) = 1 - \theta_1 B - \theta_2 B^2 - ... - \theta_q B^q$ is moving average polynomial of order $q$,

$d$ s the order of differencing, $K$ is constant, $a_t$ are for each $t$ random variables with properties of white noise.

If random components $a_t$ of the model (1) are correlated in their second moments, it is possible to find out other autoregressive model of order $q$ called ARCH (q) model to describe heteroscedasticity in time $t$, $h_t$. Model for correlated squared random components was introduced by Engle [3] in the form

$$h_t = \xi + \alpha_1 a_{t-1}{}^2 + \alpha_2 a_{t-2}{}^2 + ... + \alpha_q a_{t-q}{}^2 \qquad (2)$$

where

$\xi > 0$ and $\alpha_j \geq 0$ for $j$ = 1, 2, ..., $q$. ARCH effect is usually tested by means of Lagrange multiplier test [4].

## 3. Results of the analysis and estimation of the models

### 3.1 AR(2) – ARCH(3) model

To estimate an ARCH model for variable *ri_ratsfond* from Picture 3 we shall use following steps:

1) Estimation of the mean equation for *ri_ratsfond*

$$ri\_asfond_t = c + a_t \quad \text{for t = 2, 3,..., 238} \qquad (3)$$

$$\underset{(0{,}000277)}{ri\_rasfond_t = 1{,}000195}$$ with standard error in parenthesis.

Test of residuals of equation (3) have showed the statistically significant second coefficient of not only autocorrelation but partial autocorrelation function as well. The model (3) was widespread by means of lagged variable *ri_rastfond* to model (4) and estimated with following results

$$ri\_asfond_t = c + ri\_rastfond_{t-2} + a_t \tag{4}$$

$$ri\_asfond_t = \underset{(0{,}0639)}{0{,}78562} + \underset{(0{,}0638)}{0{,}214277} ri\_rastfond_{t-2}$$

$R^2 = 0{,}046$; D-W=1,95; Akaike = – 8,107 and Schwartz criterion = – 8,078. Residuals of model (4) are not correlated, but their squares are, so ARCH effect is present with statistically significant Lagrange multiplier test LM = 21,12 for four lags. The third lagged coefficient of the squared residuals of the model (4) was statistically significant, so ARCH (3) model would be estimated together with the model (4).

2) Estimation of the variance equation for *ri_rastfond*

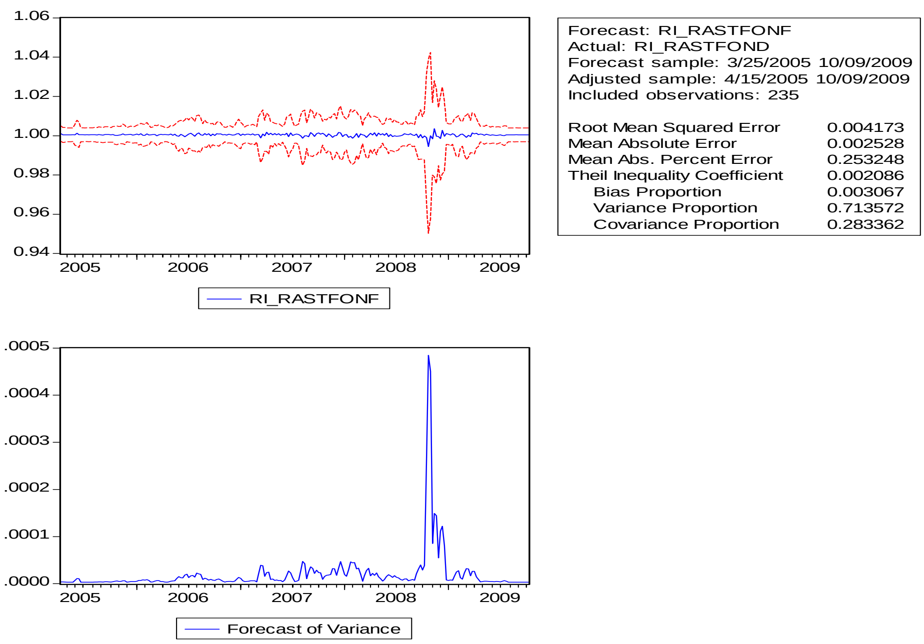
The new estimation of the mean model (4) together with variance ARCH (3) model is

$$ri\_rasfond_t = \underset{(0{,}054271)}{0{,}827695} + \underset{(0{,}054251)}{0{,}172683} ri\_rastfond_{t-2}$$

$$h_t = \underset{(8{,}10E-07)}{3{,}01E-06} + \underset{(0{,}069161)}{0{,}205526}\hat{a}_{t-1} + \underset{(0{,}133384)}{0{,}401648}\hat{a}_{t-2} + \underset{(0{,}053881)}{0{,}379438}\hat{a}_{t-3}$$

with

$R^2 = 0{,}041$; D-W=1,95; Akaike info criterion = – 8,157; Schwartz criterion = – 8,487 and Theil inequality coefficient = 0,002086 . The Picture 5 shows predicted values of *ri_rastfond* together with 95 % confidence interval of variance. Given its statistical characteristics we know, that mean absolute percentage error MAPE = 0,25 %.




**Picure 5** Predicted values of model AR(2) + ARCH(3) of the variable *ri_rastfond*.

### 3.2. ADL(2) +ARCH(3)

We have tested hypothesis whether the average world's price of crude oil could influenced the growth rates in the average value of pension units of growth pension funds in Slovakia and how many weeks ago this influence would start. By means of Granger causality tests (output is in Table 1), it was found out, that lag of 3 weeks is statistically significant at 5 % level of significance.

**Table 1 Testing Granger causality test for variables *ropa* and *ri_rastfond***

**Pairwise Granger Causality Tests**

Sample: 3/25/2005 10/15/2009

Lags: 3

| Null Hypothesis: | Obs | F-Statistic | Probability |
|---|---|---|---|
| ROPA does not Granger Cause RI_RASTFOND | 234 | 4,67888 | 0.00342 |
| RI_RASTFOND does not Granger Cause ROPA | | 2.46317 | 0.06327 |

For this reason ADL (3) mean model for exogenous variable *ropa* was suggested of the form

$$ri\_rasfond_t = c + ropa_{t-3} + a_t \tag{5}$$

together with ARCH (3) model of its residuals.

1) Estimated mean equation for variable *ri_rastfond* with lagged exogenous variable *ropa* is

$$\begin{aligned} ri\_rasfond_t &= 1{,}002107 - 2{,}55E - 05 ropa_{t-2} \\ &\quad\ (0{,}00036) \quad (4{,}95\text{E-}06) \end{aligned}$$

2) Estimated variance equation for residuals of model (5) is

$$\begin{aligned} h_t &= 1{,}60E - 06 + 0{,}364349\hat{a}_{t-1} + 0{,}407964\hat{a}_{t-2} + 0{,}665751\hat{a}_{t-3} \\ &\quad (2{,}94\text{E-}07) \quad (0{,}092433) \qquad (0{,}127788) \qquad (0{,}123668) \end{aligned}$$

with

$R^2 = 0{,}022$; D-W=2,04; Akaike = – 8,58 and Schwartz criterion = – 8,49 and Theil inequality coefficient 0,001207 and MAPE = 0,26 %.

## Conclusion

On the basis of statistical analysis of the growth rates of the average value of pension units of growth pension funds in Slovakia it was found out, that they could be modeled by AR(2) and also by means of ADL(3) with the exogenous variable of average price of crude oil. The squared residuals of these models are correlated with lag of 3, so variance equation is given ARCH model.

We have been surprised that results of both these modeled combinations are quite similar not only in their Theil inequality coefficient but also in their mean characteristics of their residuals.

## Acknowledgement

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# MARKOV DECISION CHAINS IN DISCRETE- AND CONTINUOUS-TIME; A UNIFIED APPROACH

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**Abstract:** In this note we consider Markov decision chains with finite state space in discrete- and continuous-time setting for discounting and averaging optimality criteria. Connections between discounted and averaging optimality along with uniformization methods are employed for producing bounds on optimal discounted and average rewards.

**Keywords:** discrete-time and continuous-time Markov decision chains, discounted and averaging optimality, connections between discounted and averaging models, uniformization.

## 1. Introduction

In this note, we consider Markov reward processes with finite state and action spaces in discrete- and continuous-time setting. Attention will be primarily focused on connections and similarity between discrete- and continuous-time Markov decision chains useful for finding optimal discounted and averaging control policies. According to the best of our knowledge, in the existing literature generating lower and upper bounds in averaging and discounted optimality was studied only for discrete-time models; in the present note we show how these methods also work in the continuous-time case. Also uniformization methods will be employed for producing bounds on optimal discounted and average rewards.

**2. Notations and Preliminaries**

We consider Markov decision processes with finite state space $I=\{1,2,\dots,N\}$ both in discrete- and continuous-time. In the discrete-time case, we consider Markov decision chain $X^{\mathrm{d}}=\{X_n, n=0,1,\dots\}$ with finite state space $I=\{1,2,\dots,N\}$, and finite set $A_i=\{1,2,\dots,K_i\}$ of possible decisions (actions) in state $i\in I$. Supposing that in state $i\in I$ action $a\in A_i$ is selected, then state $j$ is reached in the next transition with a given probability $p_{ij}(a)$ and one-stage transition reward $r_{ij}$ will be accrued to such transition.

In the continuous-time setting, the development of the considered Markov decision process $X^{\mathrm{c}}=\{X(t), t\ge 0\}$ (with finite state space $I$) over time is governed by the transition rates $q(j\,|\,i,a)$, for $i,j\in I$, depending on the selected action $a\in A_i$. For $j\ne i$ $q(j\,|\,i,a)$ is the transition rate from state $i$ into state $j$, $q(i\,|\,i,a)=\sum_{j\in I,\, j\ne i} q(j\,|\,i,a)$ is the transition rate out of state $i$. As concerns reward rates, $\tilde{r}(i)$ denotes the rate earned in state $i\in I$, and $\tilde{r}(i,j)$ is the transition rate accrued to a transition from state $i$ to state $j$.

A (Markovian) policy controlling the decision process is given either by a sequence of decision at every time point (discrete-time case) or as a piecewise constant right continuous function of time (continuous-time case). In particular, for discrete-time models policy controlling the chain, $\pi=(f^0, f^1,\dots)$, is identified by a sequence of decision vectors $\{f^n, n=0,1,\dots\}$ where $f^n\in A\equiv A_1\times\dots\times A_N$ for every $n=0,1,2,\dots$, and

$f_i^n \in A_i$ is the decision (or action) taken at the $n$th transition if the chain $X^{\mathrm{d}}$ is in state $i$. Policy which selects at all times the same decision rule, i.e. $\pi \sim (f)$, is called stationary; $P(f)$ is transition probability matrix with elements $p_{ij}(f_i)$.

Similarly, for the continuous-time case policy controlling the chain, $\pi = f^t$, is a piecewise constant, right continuous vector function where $f^t \in A \equiv A_1 \times \ldots \times A_N$, and $f_i^t \in A_i$ is the decision (or action) taken at time $t$ if the process $X(t)$ is in state $i$. Since $\pi$ is piecewise constant, for each $\pi$ we can identify time points $0 < t_1 < t_2 \ldots < t_i < \ldots$ at which the policy switches; we denote by $f^{(i)} \in F$ the decision rule taken in the time interval $(t_{i-1}, t_i]$. Policy which takes at all times the same decision rule, i.e. $\pi \sim (f)$, is called stationary; $Q(f)$ is transition rate matrix with elements $q(j \mid i, f_i)$.

The more detailed analysis requires to consider the discrete- and continuous-time case separately. In this note we make the following assumption.

**Assumption A.** There exists state $i_0 \in I$ that is accessible from any state $i \in I$ for every $f \in A$, i.e. for every $f \in A$ the transition probability matrix $P(f)$ or the transition rate matrix $Q(f)$ is *unichain* (i.e. $P(f)$ or $Q(f)$ have no two disjoint closed sets).

### 2.1. Discrete-Time Case

We denote by $P(f) = [p_{ij}(f_i)]$ the $N \times N$ transition matrix of the chain $X^{\mathrm{d}}$. Recall that the limiting matrix $P^*(f) = \lim_{m\to\infty} m^{-1} \sum_{n=0}^{m-1} P^n(f)$ exists; in case that the chain is aperiodic even $P^*(f) = \lim_{n\to\infty}(P(f))^n$. In particular, if $P(f)$

is unichain (i.e. $P(f)$ contains a single class of recurrent states) the rows of $P^*(f)$, denoted $p^*(f_i)$, are identical. Obviously, $r_i(f_i) = \sum_{j=1}^{N} p_{ij}(f_i) r_{ij}$ is the expected one-stage reward obtained in state $i \in I$ and $r(f)$ denotes the corresponding $N$-dimensional column vector of one-stage rewards. Then $[P(f)]^n \cdot r(f)$ is the (column) vector of rewards accrued after $n$ transitions; its $i$th entry denotes expectation of the reward if the process $X^{\mathrm{d}}$ starts in state $i$.

Let $\xi^n_{X_0}(\pi) = \sum_{k=0}^{n-1} r_{X_k}(f^k_{X_k})$ (resp. $\xi^{\beta,n}_{X_0}(\pi) = \sum_{k=0}^{n-1} \beta^k r_{X_k}(f^k_{X_k})$) be the (random) total reward (resp. total $\beta$-discounted reward) received in the $n$ next transitions of the considered Markov chain $X$ if policy $\pi = (f^n)$ is followed and the chain starts in state $X_0$. Then for the total expected reward $v^n_i(\pi)$ and for the total expected discounted reward $v^{\beta,n}_i(\pi)$ we have $v^n_i(\pi) = \mathrm{E}^\pi_i \sum_{k=0}^{n-1} r_{X_k}(f^k_{X_k})$, $v^{\beta,n}_i(\pi) = \mathrm{E}^\pi_i \sum_{k=0}^{n-1} \beta^k r_{X_k}(f^k_{X_k})$ ($\mathrm{E}^\pi_i$ is the expectation if the process starts in state $i$ and policy $\pi$ is followed). Then for the vectors of total rewards $v^n(\pi)$ and total discounted rewards $v^{\beta,n}(\pi)$ we get

$$v^n(\pi) = \sum_{k=0}^{n-1} \prod_{j=0}^{k-1} P(f^j)\, r(f^k), \qquad \text{resp.} \qquad v^{\beta,n}(\pi) = \sum_{k=0}^{n-1} \prod_{j=0}^{k-1} \beta^k\, P(f^j)\, r(f^k). \tag{1}$$

For $n \to \infty$ elements of $v^n(\pi)$ (resp. $v^{\beta,n}(\pi)$) can be typically infinite (resp. bounded by $M/(1-\beta)$ where $M = \max_i \max_k r_i(k)$). Following stationary policy $\pi \sim (f)$ for $n$ tending to infinity there exist vectors of average rewards per transition, denoted $g(f)$ (with elements $g_i(f)$ bounded by $M$), and vector of total discounted rewards, denoted $v^\beta(f)$, with elements $v_i^{\beta}(f)$ being the discounted reward if the process starts in state $i$, where ($I$ denotes the identity matrix)

$$g(f) := \lim_{n\to\infty} \frac{1}{n} v^n(f) = P^*(\pi)\, r(f) \qquad (2)$$

$$v^{\beta}(f) := \sum_{k=0}^{\infty} [\beta\ P(f)]^k\ r(f) = [\,I - \beta\, P(f)]^{-1}\ r(f) = r(f) + \beta\ v^{\beta}(\pi). \quad (3)$$

Let for arbitrary policy $\pi = (f^n)$ $\hat{v}^{\beta} := \sup_{\pi} v^{\beta}(\pi)$, $\quad \hat{g} := \sup_{\pi} \liminf_{n\to\infty} \frac{1}{n} v^n(\pi)$

where $\hat{v}_i^{\beta}$, resp. $\hat{g}_i$ (the $i$th element of $\hat{v}^{\beta}$, resp. of $\hat{g}$) is the maximal $\beta$-discounted reward, resp. maximal average reward, if the process starts in state $i \in I$. Moreover, under Assumption A for every stationary policy $\pi$ □ $(f)$ the vector $g(f)$ is a constant vector with elements $\overline{g}(f)$ equal to $p^*(\pi)\ r(f)$.

The following facts are well-known to workers in stochastic dynamic programming (see e.g.[1,4,8,9,13]).

**Fact 1.** (i) There exists decision vector $\hat{f}^{\beta} \in A$ along with (column) vector $\hat{v}(\beta) = v(\hat{f}^{\beta}, \beta)$, being the unique solution of

$$v^{\beta}(f) = \max_{f\in A}[\, r(f) + \beta\, P(f)\ v^{\beta}(f)]. \qquad (4)$$

In particular, for elements of $\hat{v}^{\beta}$, denoted $\hat{v}^{\beta}$, we can write

$$\hat{v}_i^{\beta} = \max_{a\in A(i)}[r_i(a) + \beta \sum_{j\in I} p_{ij}(a)\ \hat{v}_j^{\beta}\,] = r_i(\hat{f}_i^{\beta}) + \beta \sum_{j\in I} p_{ij}(\hat{f}_i^{\beta})\hat{v}_j^{\beta}\ . \qquad (5)$$

(ii) If Assumption A holds there exists decision vector $\hat{f} \in A$ along with (column) vectors $\hat{w} = w(\hat{f})$ and $\hat{g} = g(\hat{f})$ (constant vector with elements $\overline{g}(f) = p^*(\hat{f}) r(\hat{f})$) being the solution of

$$w(f)+g(f)=\max_{f\in A}[\,r(f)+P(f)\ w(f)] \tag{6}$$

where $w(\hat{f})$ is unique up to an additive constant, and unique under the additional normalizing condition $P^*(f)\ w(f)=0.$ In particular, for elements of $\hat{g}=g(\hat{f})$, and $\hat{w}=w(\hat{f})$, denoted $\overline{g}$ and $\hat{w}_i$, we can write

$$\hat{w}_i+\overline{g}=\max_{a\in A(i)}[r_i(a)+\sum_{j\in I}p_{ij}(a)\hat{w}_j]=r_i(\hat{f}_i)+\sum_{j\in I}p_{ij}(\hat{f}_i)\hat{w}_j. \tag{7}$$

## 2.2. Continuous-Time Case

Let for $f\in F$ $Q(f)=[q_{ij}(f_i)]$ be the $N\times N$ matrix whose $ij$ th element $q_{ij}(f_i)=q(j\,|\,i,f_i)$ for $i\neq j$ and for the $ii$ th element we set $q_{ii}(f_i)=-q(i\,|\,i,f_i)$. The sojourn time of the considered process $X^{\mathrm{c}}$ in state $i\in I$ is exponentially distributed with mean value $q(i\,|\,i,f_i)$. Hence the expected value of the reward rate obtained in state $i\in I$ equals $r_i(f_i)=q(i\,|\,i,f_i)\tilde{r}(i)+\sum_{j\in I,\,j\neq i}q(j\,|\,i,f_i)\tilde{r}(i,j)$ and $r(f)$ is the (column) vector of reward rates at time $t$.

For any policy $\pi=(f^t)$ the accompanying set of transition rate matrices $\{Q(f^t),t\geq 0\}$ determines a continuous-time (in general, nonstationary) Markov process.

Let $P(\cdot,\cdot,\pi)$ be the $N\times N$ matrix of transition functions associated with Markov chain $X^{\mathrm{c}}$, i.e. for each $0\leq s\leq t$ the $ij$ th element of $P(s,t,\pi)$, denoted $P_{ij}(s,t,\pi)$, is the probability that the chain is in state $j$ at time $t$ given it was in state $i$ at time $s$ and policy $\pi$ is followed. Obviously, $P(s,t,\pi)=P(s,u,\pi)\,P(u,t,\pi)$ for each $0\leq s\leq u\leq t$. The values $P(s,t,\pi)$ are absolutely continuous in $t$ and satisfy the system of differential equations (except possibly where the piecewise constant policy switches)

$$\frac{\partial P(s,t,\pi)}{\partial t} = P(s,t,\pi)Q(f^t), \qquad \frac{\partial P(s,t,\pi)}{\partial s} = -Q(f^s)P(s,t,\pi) \tag{8}$$

where $P(s,s,\pi) = I$ ($I$ is an $N \times N$ unit matrix). In what follows it will be often convenient to let $P(t,\pi) = P(0,t,\pi)$. By (8) we then immediately get for any $t \geq 0$

$$\frac{\mathrm{d}P(t,\pi)}{\mathrm{d}t} = P(t,\pi)Q(f^t) \Longleftrightarrow P(t,\pi) = I + \int_0^t P(u,\pi)Q(f^u)\mathrm{d}u \tag{9}$$

In particular, for $\pi$ □ $(f)$ we have

$$P(t,\pi) = \exp[Q(f)t] = \sum_{k=0}^{\infty} \frac{1}{k!}(Q(f)t)^k. \tag{10}$$

It is well known that for any stationary policy $\pi = (f)$ there exists $\lim_{t\to\infty} P(t,\pi) = P^*(\pi)$ and, moreover, that for any $t \geq 0$ it holds $P(t,\pi)P^*(\pi) = P^*(\pi)P(t,\pi) = P^*(\pi)$ along with

$P^*(\pi)P^*(\pi) = P^*(\pi), \quad Q(f)P^*(\pi) = P^*(\pi)Q(f) = 0$.

If policy $\pi = (f^t)$ is followed then for the vector of total $\rho$-discounted rewards $V^{t,\rho}(\pi)$ (with discount factor $\rho > 0$) it holds

$$V^T(\pi) = \int_0^T P(t,\pi)r(f^t)\mathrm{d}t,$$

$$V^{T,\rho}(\pi) = \int_0^T \mathrm{e}^{-\rho t} P(t,\pi)r(f^t)\mathrm{d}t \tag{11}$$

(the $i$th element of $V^{T,\rho}(\pi)$ denoted $V_i^{T,\rho}(\pi)$ is the reward is the process starts in state $i$).

Following stationary policy $\pi$ □ $(f)$ for $T$ tending to infinity there exist vectors of average rewards per transition, denoted $G(f)$ (with bounded entries $G_i(f)$ and vector of total discounted rewards, denoted $V^{\rho}(f)$, such that

$$G(f) := \lim_{T\to\infty} \frac{1}{T} \int_0^T P(t,\pi) r(f) \mathrm{d}t = P^*(\pi) r(f) \tag{12}$$

$$V^\rho(f) := \lim_{T\to\infty} \int_0^T \mathrm{e}^{-\rho t} P(t,\pi) r(f) \mathrm{d}t = \rho^{-1} [r(f) + Q(f) V^\rho(f)] \tag{13}$$

The following facts are well-known to workers in stochastic dynamic programming (see e.g.
[1,2,4,8,9,13]).

**Fact 2.** (i) There exists decision vector $\hat{f}^{(\rho)} \in A$ along with (column) vector $\hat{V}^\rho = V^\rho(\hat{f}^{(\rho)}),$ being the unique solution of

$$\rho V^\rho(f) = \max_{f\in A} [r(f) + Q(f) V^\rho(f)]. \tag{14}$$

In particular, for elements of $\hat{V}^\rho$, denoted $\hat{V}_i^\rho$, we can write

$$\rho \hat{V}_i^\rho = \max_{a\in A(i)} [r_i(a) + \sum_{j\in I} q_{ij}(a) \hat{V}_j^\rho] = r_i(\hat{f}_i^{(\rho)}) + \sum_{j\in I} q_{ij}(\hat{f}_i^{(\rho)}) \hat{V}_j^\rho \ . \tag{15}$$

(ii) If Assumption A holds there exists decision vector $\hat{f} \in A$ along with (column) vectors $\hat{W} = W(\hat{f})$ and $\hat{G} := G(\hat{f}) = P^*(\hat{f}) r(\hat{f})$ (constant vector with elements $\bar{G}(f) = p^*(\hat{f}) r(\hat{f})$) being the solution of

$$G(f) = \max_{f\in A} [r(f) + Q(f) W(f)] \tag{16}$$

where $W(\hat{f})$ is unique up to an additive constant, and unique under the additional normalizing condition $P^*(f) W(f) = 0.$

## 3. Discounted and Averaging Optimality Equations

In this section we discuss connections between optimality equations for discounted and undiscounted models using a simple transformation of discounted model into the undiscounted unichain case. Furthermore, we indicate how continuous-time models can be transformed to discrete state models. The results are adapted from [12] and present a unified approach to various results scattered in the literature (see e.g. [3, 5, 6, 7, 10, 11, 14]).

**Theorem 1.** The discounted maximal (resp. current) total reward if the process starts in state $\ell$ equals the maximal (resp. current) average reward of the Markov reward process if

*For the discrete-time case.* The transition probability matrix $P(f)$ in (4) is replaced by the transition probability matrix $P^{(\ell)}(f) := \beta P(f) + A^{(\ell)}$ where $A^{(\ell)}$ is a square matrix such that the $\ell$th column is equal to $(1-\beta)$, and elements of the remaining columns equal zero, and the $\ell$th element of vector $w(f)$ equals zero. Then $\hat{v}_i^{\beta}$ equals elements of $(1-\beta)^{-1}\hat{g}$.

*For the continuous-time case.* The transition rate matrix $Q(f)$ in (14) is replaced by the transition rate matrix $Q^{(\ell)}(f) := \rho^{-1}Q(f) - I + B^{(\ell)}$ where $B^{(\ell)}$ is a square matrix such that only the $\ell$th column is non-null with elements equal to unity, and the $\ell$th element of vector $W(f)$ (unique up to additive constant) equals zero. Then $\hat{V}_{\ell}^{\rho}$ equals elements of $\hat{G}$.

*Proof.* Obviously, results for the current policy follow immediately from results for optimal policy if we shrink the set of feasible policies to a single policy.

*For the discrete-time case* *(I* is an identity matrix, $e$ denotes unit column vector*)*

$v^{\beta} = \max_{f \in A}[r(f) + \beta P(f)v^{\beta}] \Leftrightarrow (1-\beta)v_{\ell}^{\beta}\, e =$
$\max_{f \in A}[r(f) + (\beta P(f) - I)(v^{\beta} - v_{\ell}^{\beta}\, e)] \Leftrightarrow$
$w^{(\ell)} + g\, e = \max_{f \in A}[r(f) + P^{(\ell)}(f)\, w^{(\ell)}]$ where

$g := (1-\beta)v_{\ell}^{\beta}$, $w^{(\ell)} := v^{\beta} - v_{\ell}^{\beta}\, e$, $P^{(\ell)}(f) = \beta P(f) + A^{(\ell)}$ (observe that $P^{(\ell)}(f)$ is a stochastic matrix and that $w_{\ell}^{(\ell)} = 0$)

*For the continuous-time case*

$\rho V^{\rho} = \max_{f \in A}[r(f) + Q(f)V^{\rho}] \Leftrightarrow \quad 0 = \max_{f \in A}[\rho^{-1}r(f) + (\rho^{-1}Q(f) - I)V^{\rho}] \Leftrightarrow$
$V_{\ell}^{\rho}\, e = \max_{f \in A}[\rho^{-1}r(f) + \rho^{-1}Q(f)V^{\rho} - [V^{\rho} - V_{\ell}^{\rho}\, e]] \Leftrightarrow$
$V_{\ell}^{\rho}\, e = \max_{f \in A}[\rho^{-1}\, r(f) + \rho^{-1}Q(f)[V^{\rho} - V_{\ell}^{\rho}\, e] - [V^{\rho} - V_{\ell}^{\rho}\, e]]$
Then for $G := V_{\ell}^{\rho}\, e$ and $W^{(\ell)} := V^{\rho} - V_{\ell}^{\rho}\, e$ we can write

$$G = \max_{f \in A}[\rho^{-1}\, r(f) + \rho^{-1}Q(f)W^{(\ell)} - W^{(\ell)}]$$
$$= \max_{f \in A}[\rho^{-1}\, r(f) + \rho^{-1}Q^{(\ell)}(f)W^{(\ell)}]$$

for $Q^{(\ell)}(f) := \rho^{-1}Q(f) - I + B^{(\ell)}$
(observe that $Q^{(l)}(f) := \rho^{-1}Q(f) - I + B^{(\ell)}$ is a transition rate matrix and that $W_{\ell}^{(\ell)} = 0$).

**Theorem 2.** The continuous-time maximal (resp. current) average reward being the solution of (16) equals the discrete-time maximal (resp. current) average reward if in the optimality equation (6) we set

$$P(f) := B^{-1}Q(f) + I \quad \text{where} \quad B > \max_{f \in A, i \in I} \sum_{j \in I, j \neq i} q_{ij}(f). \tag{17}$$

Then $g(f) = G(f)$ and $w(f) = BW(f)$.

*Proof.* First observe that element of $P(f) := B^{-1}Q(f) + I$ are nonnegative, nongreater than unity and all row sums equal unity. From (16) we get

$$G(f) = \max_{f \in A}[r(f) + Q(f)W(f)] \Leftrightarrow$$

$$W(f) + B^{-1}G(f) = \max_{f \in A}[B^{-1}r(f) + [B^{-1}Q(f) + I]W(f)] \Leftrightarrow$$

$$BW(f) + G(f) = \max_{f \in A}[r(f) + [B^{-1}Q(f) + I]BW(f)] \Leftrightarrow$$

$$w(f) + g(f) = \max_{f \in A}[r(f) + P(f)w(f)]$$

**Theorem 3**. The vector of continuous-time maximal (resp. current) $\rho$-discounted reward being the solution of (14) equals the discrete-time maximal (resp. current) $\beta$-discounted reward if the optimal equation (4) takes on the following form

$$v(f) = \max_{f}[B^{-1}r(f) + [B^{-1}(Q(f) - \rho I) + I]v(f)]$$

where $B > \max_{f \in A, i \in I} \sum_{j \in I, j \neq i} q_{ij}(f)$. (18)

*Proof.* First observe that elements of the matrix $\tilde{P}(f) := [B^{-1}(Q(f) - \rho I) + I]B^{-1}$ are nonnegative, nongreater than unity and all row sums equal $1 - \beta$. From (14) we get

$$\rho V = \max_{f \in A}[r(f) + Q(f)V] \Leftrightarrow$$

$$V(f) = \max_{f \in A}[B^{-1}r(f) + [B^{-1}(Q(f) - \rho I) + I]V(f)] \Leftrightarrow$$

$$v(f) = \max_{f \in A}[B^{-1}r(f) + [B^{-1}(Q(f) - \rho I) + I]v(f)]$$

## 4. Conclusions

In this note we focused attention on optimality equations for discrete- and continuous time Markov decision chains if discounted and averaging optimality criteria are considered. Using a suitable data transformation we shown connections between discounted and averaging optimality equations, and using the uniformization technique also connections between discrete- and continuous-time models.

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# MATHEMATICS OF POLITICSECONOMICS METHODOLOGY IN POLITICAL SCIENCE

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**Abstract:** In this paper the concept of politometrics is proposed, defined in a similar way as Ragnar Frisch in 1932 had defined the econometrics: as the discipline summarizing application of mathematical and statistical techniques to political science problems and theories. Different possible topics of politometrics are discussed, such as models of voting, measures of influence in committee systems and regression models of socio-demographic determinants of voters' behavior. Relevant research agenda is briefly outlined.

**Keywords:** economics, mathematics, politics, power, voting

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## 1. Economics, political science and mathematics

The use of mathematics in the social sciences is expanding both in breadth and depth at an increasing rate. It has made its way from economics into the other social sciences, often accompanied by the same controversy that raged in economics in the 1950's. The reasons for this expansion are several: "First, mathematics makes communication between researchers succinct and precise. Second, it helps make assumptions and models clear; this bypasses arguments in the field that are a result of different implicit assumptions. Third, proofs are rigorous, so mathematics helps avoid mistakes in the literature. Fourth, its use often provides more insights into the models. And finally, the models can be applied to different contexts without repeating the analysis, simply by renaming the symbols."[1]

In this paper we follow the Duncan Black's idea that "Economics and Political Science are the same in kind: that when we do eventually obtain a 'satisfactory' Political Science, it will have the same distinguishing marks as Walras' Elements or Pareto's Manuel, or perhaps Marshall's Principles, with the admixture of the rigorously formal and the descriptive treatment – rather than those of the existing texts in Politics. And the core of the treatment, we hold, will consists of a set of formal or mathematical propositions".[2]

First Nobel Prize Laureate for economics (1969), Norwegian economist Ragnar Frisch, introduced in 1932 concept of econometrics as "the application of mathematical and statistical techniques to economic problems and theories". During last 70 years econometrics evolved to be one of the fundamental instruments of economic analyses. In a broader sense (as was originally interpreted by Ragnar Frisch) econometrics formulate mathematical models of economic processes and using observable data discovers directly unobservable properties, verifies propositions or conclusions derived from abstract model analysis. In a narrow sense econometrics is an instrument for empirical testing of hypotheses of economic theory.

While the focus of economics is on human behavior on the market place, characterized by data such as GDP, inflation, unemployment, income, consumption, investments, savings, trade, etc., in political science we study human behavior in the public arena, outside of the market, characterized by electoral preferences, political behavior in representative bodies, behavior of

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[1] Schofield, N. (2004), Mathematical Methods in Economics and Social Sciences, Springer, Berlin, Heidelberg, New York.

[2] Quoted from Black D., The Theory of Committees and Elections, revised second edition, edited by Iain McLean, Alistair McMillan and Burt L. Monroe, Kluwer Academic

central, territorial and local governments etc. In both cases we have some observable data, time series, provided by statistical service, sample studies, electoral statistics etc. It was economics which started to understand that traditional dichotomy between "homo oeconomicus" and "homo politicus" is counterproductive and that it is not possible to understand and explain economic phenomena without study of political behavior: in a society where 40% of GDP is redistributed by different decision making bodies, where provision of public goods is regulated by various councils, committees, parliaments, where positive and negative externalities lead to market failures, the doctrine of "self-regulated market place" and "invisible hand of competition" does not provide satisfactory answers to appealing questions. Thus, approximately since 1948, within the framework of economic sciences new disciplines emerged, studying the problems of collective choice, bureaucracy behavior, rent seeking, voting behavior, institutions etc. In this respect we can speak about elements of application of economics methodology to political sciences. This approach extends power and deepness of economic analyses and provides new interesting theoretical and empirical results. It is interesting, that among Nobel Prize Laureates for economics it is possible to find outstanding scientists representing this orientation in economics: Kenneth J. Arrow (1972), James M. Buchanan (1986), John Nash, John Harsanyi and Richard Selten (1994), Amartya K. Sen (1998).

The question is: why political sciences, using similar data, should not use similar methodology? Concept of politometrics, proposed in this paper, can be defined in the same way as Ragnar Frisch defined econometrics: the application of mathematical and statistical techniques to political problems and theories.

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In this paper we try to illustrate in a very simple way possible subjects of politometrics, such as modeling of political processes, analyzing electoral systems, measuring power in committees, explaining voters' behavior. We also shortly outline some topics of the relevant research agenda.

First time the concept of politometrics was introduced in Turnovec (2003). In recent literature the comprehensive treatment of the mathematics applications in political science see e.g. in Brams (2008), Taylor and Pacelli (2008), Schofield (2004).

## 2. Voting as an aggregation of individual preferences

By voting we mean the following pattern of collective choice: There is a set of alternatives and a group of individuals. Individual preferences over the alternatives are exogenously specified and are supposed to be orderings. The group is required to choose an alternative on the basis of stating and aggregating of all individual preferences, or to produce a ranking of alternatives from the most preferred to the least preferred.

To show that problems with voting are not as simple as one can expect, let us start with almost trivial example.

**Example 1:** Consider 3 candidates, A = {x, y, z}, and 9 voters, N = {1, 2, 3, 4, 5, 6, 7, 8, 9} with a preference profile given in the Table 1.

Table 1

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| z | z | z | z | x | x | x | y | y |
| x | y | y | y | y | y | y | x | x |
| x | x | x | x | z | z | z | z | z |

We can produce a pair-wise comparisons matrix with entries indicating how many voters prefer a "row" candidate to the "column candidate" (Table 2).

Table 2

| | x | y | z |
|---|---|---|---|
| x | - | 3 | 5 |
| y | 6 | - | 5 |
| z | 4 | 4 | - |

Let us apply several standard voting procedures to this situation:

THE CONDORCET' VOTING PROCEDURE: The candidate is chosen if he is not defeated by a strict majority by any other candidate. In our case the candidate y is selected and we receive the ordering y P x P z (by *a* P *b* we denote that a candidate *a* is collectively, by voting, preferred to candidate *b*).
PLURALITY VOTING PROCEDURE: The candidate who is preferred by most of the voters to all other candidates is selected. In our case this voting procedure selects candidate z and generates the ordering z P x P y.
PLURALITY WITH RUN-OFF: If no alternative receives a majority of votes on the first ballot, the top two vote getters are considered and the candidate who receives majority in the run-off is selected. In our case the candidate x is selected and we receive the ordering x P z P y.
We can see that using three different voting procedures we can get three completely different results of voting based on the same individual preferences.

The problem of voting has two aspects: individual decision of the voter (his choice among the alternatives) and the counting the votes, evaluation of results. Social choice theory investigates two aspects of voting rules and procedures: Individual voting behavior expresses individual preferences of the voter. The method of counting should guarantee an aggregation of individual preferences into something called social preferences. The key question is: how to aggregate individual preferences by some intuitively acceptable and not contradictory way into social preferences?

Selected problems:

a) Democratic legitimacy, existence of not-contradictory algorithms of aggregating individual preferences (Arrow's impossibility theorem puzzle – collective choice is either not transitive or dictatorial).

b) Manipulation by strategic voting: is it possible for an individual or a group of individuals to benefit by misrepresentation of their preferences? (Gibbard-Satterthwaite theorem saying that any voting procedure is either manipulable or dictatorial). Information complexity of manipulation. Rational voters (voting by sincere preferences), irrational voters (voting randomly) and sophisticated voters (voting strategically) and models of their behavior.

c) Agenda manipulation and voting rules manipulation, how to influence result of voting by its institutional framework? Political districting in majority electoral systems and fair seats allocation in proportional electoral systems.

## 3. Calculus of influence – voting power in committees

Having a committee elected it makes sense to try to analyze *of distribution of power* among its members, quantitative evaluation of an influence and voting power of different members of the committee. At this introductory level we shall again illustrate the problem of power by a simple example.

**Example 2:** Distribution of votes among the parties in a committee is not a sufficient characteristic of power or influence distribution. This can be clearly seen by a simple example of the committee with 3 parties and 100 seats (see Table 3).

Table 3

| parties | seats |
|---|---|
| 1 | 49 |
| 2 | 2 |
| 3 | 49 |

With respect to the simple majority rule all three parties have the same position in the voting process (any two-parties coalition is a winning one, no single party can win). In fact, under certain circumstances (if the two large parties 1 and 3 are on the opposite sides of the political spectrum) the role of the party 2 could be essential. Quite a different situation can be observed for a qualified majority, say, 60%. In this case the party 2 has no influence on the outcomes of voting and a co-operation of parties 1 and 3 is needed for approving any bill.

It is known that a distribution of votes among the groups in a committee is not a sufficient characteristic of their voting power or an influence distribution. So called power indices are used to estimate an influence of the members of a committee as a function of a voting rule and of a structure of representation in a committee.3

The majority of proposed power indices are based on the game theoretical model of simple games in characteristic function form and on different

[3] In 1954 Lloyd Shapley and Martin Shubik published a short paper in the American Political Science Review, proposing that the Shapley value for cooperative characteristic function form games could serve as a measure of voting power in committees. In 1965 John Banzhaf proposed a new index of voting power. Since that more than twenty new definitions (with more or less sat0isfactory theoretical justification) of so called power indices have been published.

concepts of "decisiveness" of members of a committee with respect to winning coalitions. They usually express probability of members of the body to be “decisive” in a given sense.

For illustration let us consider one of the most frequently used power indices proposed by John Banzhaf, the so called Banzhaf power index. All possible winning coalitions are considered. Each of the winning coalition is analyzed and the so called "swing" voters are identified: i.e. those who by changing their vote from "yes" to "no" could change the coalition from winning to losing. The relative “voting power” of individual members is then measured by a ratio of the number of member swings to the total number of swings in the committee. Let us apply Banzhaf measure in our example (assuming always simple majority rule). Winning coalitions and swings: {A*, B*}, {A*, C*}, {B*, C*}, {A, B, C}. Each party has 2 swings out of 6, i.e. the relative power of each of them is 1/3, i.e. the vector of relative power indices equals to (1/3, 1/3, 1/3).

In fact, under certain circumstances (if the two large parties A and C are on the opposite sides of the political spectrum) the role of the party B can be essential. Let us suppose that A and C are strictly opposed blocs (they never vote together). Here we have the following winning coalitions and swings: {A*, B*}, {B*, C*}. Hence, in this case B has two swings out of four swings, while A and C has only one swing and relative voting power of the parties can be evaluated as (1/4, 1/2, 1/4).

We can introduce another assumption: let us assume that A, B, C is an ordering of the parties over some political dimension (say, left and right), and that only "ideologically connected" coalitions can be created (for example in forming a government coalition). Then we shall have the following swings: {A*, B*}, {B*, C*}, {A, B*, C}. Each of the parties A and C has one swings, while party B has three swings out of six, therefore the evaluation of relative voting power will be (1/5, 3/5, 1/5). We can see that even simple measure of power is flexible enough to reflect different assumptions about parties' behavior.

Selected problems:

d) Binary (YES-NO) voting and abstention, existing models do not consider abstention as a strategic factor, while in some cases abstention could mean NO and in some cases abstention could mean YES.

e) Fairness in voting: voting rules that guarantee distribution of power proportional to distribution of votes?

f) Coalition formation. Waiving assumption about equal probability of different voting coalitions: how to incorporate different propensity of committee members to cooperate into the model?

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## 4. Explaining voters' behavior

Parliamentary elections provide political scientists with valuable data. Statistics and econometrics propose efficient instruments for analysis of electoral results. One of the possibilities is to look for socio-economic determinants of voters' behavior. Let us demonstrate this opportunity on a simple linear model.

Let us denote by

n number of electoral districts in the country ($j = 1, 2, ..., n$), m number of political parties participating in election (i = 1, 2, ..., m),

r number of socio-demographic factors (explanatory variables), such as inflation rate, average income, age structure of population, income structure, educational structure, professional structure, rate of rural population (urbanization) etc. (k = 1, 2, ..., r),

$p_i$ percentage of votes submitted for party i in the country,

$p_{ij}$ percentage of votes submitted for party i in district j,

$x_k$ observed value of socio-demographic factor k in the country,

$x_{kj}$ observed value of socio-demographic factor k in district j,

$\beta_{ik}$ rate of influence of socio-demographic factor k on voters' decision to vote for party i (how much the percentage of votes for party i will change if the value of factor k changes by a unit).

Assuming, that the percentage of votes for a given party is a linear function of explanatory variables, we want the identify m functions

$$p_i(\mathbf{x}, \boldsymbol{\beta}_i) = \sum_{k=1}^{r} x_k \beta_{ik}$$

expressing electoral results of party i as a linear function of socio-demographic factors.

Based on observable data $p_{ij}$, $x_{kj}$, for each i we can estimate parameters $\beta_{ik}$ of the function $p_i(\mathbf{x},\boldsymbol{\beta}_i)$ as values minimizing the sum of square deviations of the function from observed values taken by electoral districts:

$$\sum_{j=1}^{n} (p_{ij} - \sum_{k=1}^{r} x_{kj} \beta_{ik})^2$$

If the number of electoral districts n is significantly greater than the number of explanatory variables r, we can use standard econometric techniques to estimate parameters $\beta_{ik}$ and to test how significant estimated parameters are. Careful selection of explanatory variables and cautious interpretation of results (estimated rates of influence of socio-demographic factors on voters' electoral choice) open a broad space for qualitative analysis explaining different aspects of voters' decision making and political parties concerns.

Selected problems:

g) How to estimate (in proportional electoral systems) switches at the two consecutive elections, what fraction of voters who voted for party i in the first election switched to party k in second election (ecological regression)?

h) Financing of political parties and efficiency of electoral campaign expenditures; does power in democracy depend on money investments?

i) Political landscape: multi-dimensionality of ideological space. Are voters voting on ideological basis?

## 5. Roots

Politometrics can find its roots in several economic disciplines, such as econometrics, public choice and social choice, constitutional economics, welfare economics. Generally one credits James Buchanan, the Nobel Prize Laureate, and Gordon Tullock as the intellectual fathers of economic studies of politics. Their book from 1962 *Calculus of Consent* remains a classic in the relevant literature. But some of the ground-stones of the public choice were laid before James Buchanan and Gordon Tullock introduced the whole area as a separate field of economic theory.

Another Nobel Prize Laureate, Kenneth Arrow, formulated the basic problem of discovering of social preferences in his work from 1951 *Social Choice and Individual Values*. In 1958 Duncan Black in his book *The Theory of Committees and Elections* and in 1957 Anthony Downs in *Economic Theory of Democracy* extended concepts of economic competition to political competition. Indian economist and 1998 Nobel Prize Laureate Amartya K. Sen contributed to the economic theory of justice (*Collective Choice and Social Welfare*, 1970). Game theory, developed in the 40's by John von Neumann and Oskar Morgenstern provided theorists with adequate methodology. In 1954 Lloyd Shapley and Martin Shubik started the branch of research focused on power analysis.

We also should not forget contribution of outstanding scientists of the 18th and 19th centuries, who were forgotten for many years and rediscovered only in the second half of 20th century, in voting theory French mathematicians Marquis de Condorcet (1743 - 1794), who was also an important political figure shortly after the Great French Revolution, and Jean Charles de Borda (1733-1799), contributed to intellectual background of democratic ideas and originated the mathematical theory of voting. In 19th century, British mathematician Charles Dodgson (1832-1898), better known as Lewis Carroll, the author of Alice in Wonderland, extended the theory of voting.

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# SOME NOTES ON CAUSALITY RELATIONS BETWEEN FINANCIAL DEVELOPMENT AND GROWTH AMONG THE VISEGRAD GROUP

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## Introduction

In the article we investigate the hypothesis of causality between financial development and economic growth in the Visegrád Group countries and we verify whether there are causal relations between analysed variables and between the economies. The verification is mainly based on Granger Causality test and VAR model (Wold Causality). The Visegrád Group (called "Visegrad Four" – V4) is a group of four countries: the Czech Republic (CZ), Hungary (HU), Poland (PL) and Slovakia (SK), which despite their similarity in level of economic development, differ in the ways they are carrying out their transformation (Inotai & Sass [1994]) and entering the Monetary Union.

There is a wide literature focused on an empirical investigation of bilateral relations between financial development and growth, though results are not straightforward. For example, according to Antonios [2010] or Muhammad & Umer [2010] the financial development has positive impact on economic growth, however Yucel [2009] shows that this relation is negative.

## Data

The research is focused on short-time relations between financial development and growth in V4 Group. Financial development can be approximated by the ratio of MFI's assets to GDP or stock exchange capitalisation to GDP.

We use in our study three variables for each country[1]:

LASSGDP[2] – logarithm of the ratio of Monetary Financial Institutions' (MFI) assets (consolidated balance sheet) to national gross domestic product [%],

CGDP[3] – logarithm of the ratio of national stock exchanges market capitalisation to national gross domestic product [%],

GDP[4] – logarithm of the real gross domestic product [mil national currency].

Data covers quarterly time series for 4 countries that belong to the Visegrád Group. All variables are in constant prices and are seasonally adjusted. Sources of statistics are mainly the International Financial Statistics online database and national stock exchange's bulletins.

We should begin our discussion with performing unit root test for each variable[5]. We carried out two tests to infer whether time series are stationary: Augmented-Dickey Fuller test (ADF) and Phillips-Perron test. Results support, at 5% significance level, the hypothesis of first order integration for each variables. Therefore for further computation we use first difference of logarithms of each variable (this can be interpreted as growth rates). The same results were obtained in case of panel data unit root test.

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[1] The letter CZ at the end of variable name stands for the Czech Republic, HU – Hungary, PL – Poland, SK – Slovakia, L – logarithm, R – constant prices 2005, D – stands for the first difference of variable, e.g. $D(X_t) = X_t - X_{t-1}$..

[2] Quarterly data for: Poland 1996-2009, Hungary 1998-2009, Czech Republic 2002-2009, Slovakia 2004-2009.

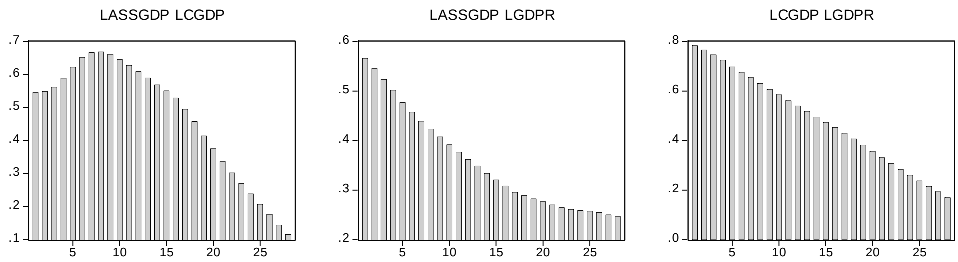
[3] Quarterly data for: Poland, Slovakia, Hungary 2001-2009, Czech Republic 2004-2009.

[4] Quarterly data for 1996-2009.

[5] Results of all test and computations are available upon a request.

The main subject of this research is to assess the existence and strength of causality between financial development and growth. First look at the cross correlation coefficients for different lags let us suppose that there is strong evidence of bidirectional correlations for levels of each variable[6]. All coefficients are positive and statistically significant (see: Fig 1).

**Fig 1. Cross correlation coefficients.**




Source: own calculations.

**Causal relations between variables**

Cross correlation analysis can give us some view on the correlation relations between variables, but it does not help in investigating the direction of such relations. Very interesting question is in what degree economic growth is influenced by financial development and what is the direction of this relation. This kind of analysis can be associated with the idea of causality. Most widely known in literature is *Granger causality* (e.g. Granger [1969], Lütkepohl [2005], p. 41-43). We can say, that one variable Granger cause other if past values of this variable help in forecasting values of the second variable. If $\Omega_t$ denotes all available in period $t$ information

[6] For 1-st differences correlation coefficients are not statistically significant.

and $z_t\left(h|\Omega_t\right)$ is optimal predictor of $z_t$ that uses information from the set $\Omega_t$, then if forecasting error for this predictor is less than any other predictor that does not include information about $x_t$, so we are better able to predict values of $z_t$ more effectively using all available information about $x_t \in \Omega_t$, we can say that $x_t$ Granger cause $z_t$.

Lütkepohl [2005, p. 48] underlines that we should be very careful in the valuation of causality. This idea can be confusing since the definition of causality is based only on the predictability features, so empirical analysis of causality is only an analysis of correlations coefficients and it has nothing in common with the causes and effects analysis in the relations between variables.

For stationary pairs of first differences of log-variables we performed Granger causality test. Table 1 summarise results for of Granger causality test for different lags (from 1 to 12 quarters) at 5% significance.

**Table 1. Granger causalities confirmed on 5% significance level.**

| Granger causality relations between variables confirmed on 5% significance level (for lags from 1 to 12) | | |
|---|---|---|
| ASS PL Granger cause CAP HU | CAP HU Granger cause CAP SK | GDP CZ Granger cause ASS PL |
| ASS PL Granger cause CAP SK | CAP HU Granger cause GDP CZ | GDP CZ Granger cause CAP SK |
| ASS PL Granger cause GDP CZ | CAP HU Granger cause GDP HU | GDP CZ Granger cause GDP HU |
| ASS PL Granger cause GDP PL | CAP HU Granger cause GDP SK | GDP CZ Granger cause GDP PL |
| ASS PL Granger cause GDP SK | CAP PL Granger cause CAP CZ | GDP CZ Granger cause GDP SK |
| CAP CZ Granger cause ASS HU | CAP PL Granger cause CAP SK | GDP HU Granger cause CAP SK |
| CAP CZ Granger cause CAP SK | CAP PL Granger cause GDP CZ | GDP HU Granger cause GDP PL |
| CAP CZ Granger cause GDP CZ | CAP PL Granger cause GDP SK | GDP HU Granger cause GDP SK |
| CAP CZ Granger cause GDP HU | CAP SK Granger cause ASS CZ | GDP PL Granger cause GDP SK |
| CAP CZ Granger cause GDP SK | GDP CZ Granger cause ASS HU | GDP SK Granger cause CAP PL |

Source: own calculations.

From the analysis of Table 1 we can see that out of 132 possible relations only 30 is statistically significant. Ratio of MFI's assets to GDP Granger causes changes in growth only in case of Poland assets - Poland influences growth in the Czech Republic, Slovakia and Poland. MFI's assets in other countries do not Granger cause any other variables. Stock exchange is very influential – changes in the Prague Stock Exchange and the Budapest Stock Exchange market capitalisation Granger cause growth in the Czech Republic, Hungary and Slovakia, the Warsaw Stock Exchange has influenced only on growth in the Czech Republic and Slovakia. In case of causality relation of growth in analysed countries, we can see that there are quite a lot of unilateral relations: the Czech Republic growth Granger cause growth in Hungary, Poland and Slovakia, Hungarian GDP growth rate Granger cause growth in Poland and Slovakia, Polish growth rate Granger causes growth only in Slovakia and Slovakia does not Granger cause any growth rates. Slovakia is the less influential country – only the Bratislava Stocks Exchange market capitalisation Granger cause changes in financial assets in the Czech Republic, and GDP growth rate in Slovakia Granger cause changes in the Warsaw Stock Exchange market capitalisation ratio. There are only two feedbacks in the system: Czech's GDP growth rate with Poland MFI's assets ratio and the Warsaw Stock Exchange market capitalisation with GDP growth rate in Slovakia.

Beside causality analysis, very important from theoretical and practical point of view, is analysis of responses of one variable for changes in other. Lee et al. [2002] emphasizes that confirmation of Granger causality does not necessarily indicate true economic reasons of relations. Important conclusions can also be obtained from impulse response analysis from multiequation systems. We can use e.g. VAR (vector autoregression) model

- system which is a generalised ADL model with endogenous variables. VAR models can be easily use for evaluation of short-time interdependencies without putting a priori theoretical assumptions about the relations. In the first step we estimated VAR(p) model: $\mathbf{y}_{i,t} = \mathbf{\Pi}^{(1)}\mathbf{y}_{i,t-1} + \mathbf{\Pi}^{(2)}\mathbf{y}_{i,t-2} + ... + \mathbf{\Pi}^{(p)}\mathbf{y}_{i,t-p} + \xi_{(m)i,t}$, where: $\mathbf{y}$ - vector of observations on all variables, $i$ - stands for county ($i = CZ, HU, PL, SK$), $t$ is time, $\mathbf{\Pi}$ - matrix of intermediate multipliers. In each equation endogenous variable is a function of its $p$ lags and $p$ lags of other variables of the system.

VAR models are very general and non-theoretical. There are a lot of advantages of using VAR models, e.g. they are easy to estimate, usually they give good forecasts, can be parsimonious parameterised (Juselius [2009], p. 14) and let for system analysis without ceteris paribus assumption and without a priori restrictions, they answer for the critic of structural multiequation models (Verbeek [2009], p. 335-336).

We use VAR model for panel data. The most important benefits of using panel data are: possibility of controlling heterogeneity, more variability of data, less colinearity. We can easily obtain more degrees of freedom. Panel data are also more useful to study dynamics of adjustment (Baltagi [2008], p. 6-7) and since most of economic relations are dynamic, (Baltagi [2008], p. 149) we use panel VAR model to verify if there are significant causality relations between variables.

We estimated VAR model for stationary first log-differences of variables. Baltagi ([2008], p. 178) puts emphasis on the importance of testing for appropriate lag length before testing causality, otherwise misleading conclusions can be obtained.

**Table 2. Lag length criteria.**

| Lag | LogL | LR | FPE | AIC | SC | HQ |
|---|---|---|---|---|---|---|
| 0 | 404 ,9995 | NA | 7 ,63e-09 | -10 ,17720 | -10 ,08722 | -10 ,14116 |
| 1 | 434 ,6735 | 56 ,34305* | 4 ,52e-09* | -10 ,70060* | -10 ,34068* | -10 ,55640* |
| 2 | 439 ,0561 | 7 ,988506 | 5 ,09e-09 | -10 ,58370 | -9 ,953846 | -10 ,33136 |
| 3 | 446 ,7657 | 13 ,46739 | 5 ,27e-09 | -10 ,55103 | -9 ,651241 | -10 ,19055 |
| 4 | 450 ,9037 | 6 ,914071 | 5 ,99e-09 | -10 ,42794 | -9 ,258214 | -9 ,959313 |

Source: own calculations.

All lag length criteria suggest one lag (see: Table 2). All inverse roots of AR characteristic polynomial for VAR(1) are inside unit circle (0,397; 0,198; 0175), so model fulfils stability condition. Chi-squared test for lag exclusion test also confirms appropriate order of VAR model $(p-value=0{,}000)$.

**Table 3. VAR Granger Causality/Block Exogeneity Wald Tests.**

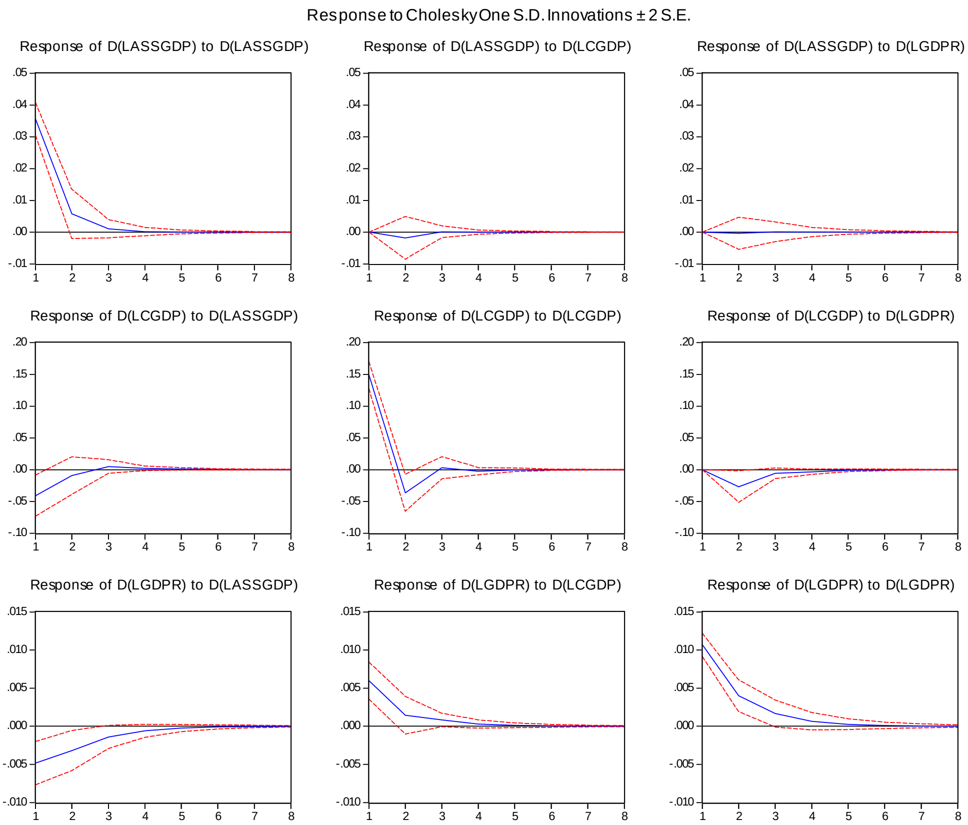
| Dependent variable: D(LASSGDP) | | | Dependent variable: D(LGDPR) | | | Dependent variable: D(LCGDP) | | |
|---|---|---|---|---|---|---|---|---|
| Excluded | Chi-sq | Prob. | Excluded | Chi-sq | Prob. | Excluded | Chi-sq | Prob. |
| D(LGDPR) | 0,017 | 0,898 | D(LASSGDP) | 1,413 | 0,235 | D(LASSGDP) | 3,003 | 0,083 |
| D(LCGDP) | 0,199 | 0,655 | D(LCGDP) | 0,379 | 0,538 | D(LGDPR) | 4,344 | 0,037 |
| All | 0,292 | 0,864 | All | 1,680 | 0,432 | All | 5,553 | 0,062 |

Source: own calculations.

According to VAR Granger Causality test (see: Table 3) we can see that only stock exchange capitalisation is caused, both by ratio of MFI's assets to GDP and by GDP growth rate.

To investigate the directions and strength of relations between variables it is very essential to perform impulse response function analysis. Impulse response analysis is conducted by calculating the responses of $j-th$ variable on the shock in $k-th$ variable. Assumption that shock is only in one variable is very controversial, if noises from different equations are correlated then shock in one variable can be associated with simultaneous shock in other variable. One way of estimation of response functions' parameters, taking into account these simultaneous correlations, is putting appropriate identifying restrictions, e.g. Cholesky decomposition or structural factorisation (SVAR). Cholesky decomposition can be non-theoretical but it needs strong assumption about recursive order of relations. Since H. Wold was the main advocate of using Cholesky decomposition, this kind of causality is called *Wold causality*. Lütkepohl ([2005], p. 61) emphasises that response functions are very sensitive for the ordering of variables. In this example Granger causality tests results will indicate the order of Cholesky decomposition.

**Fig 2. Impulse response functions.**

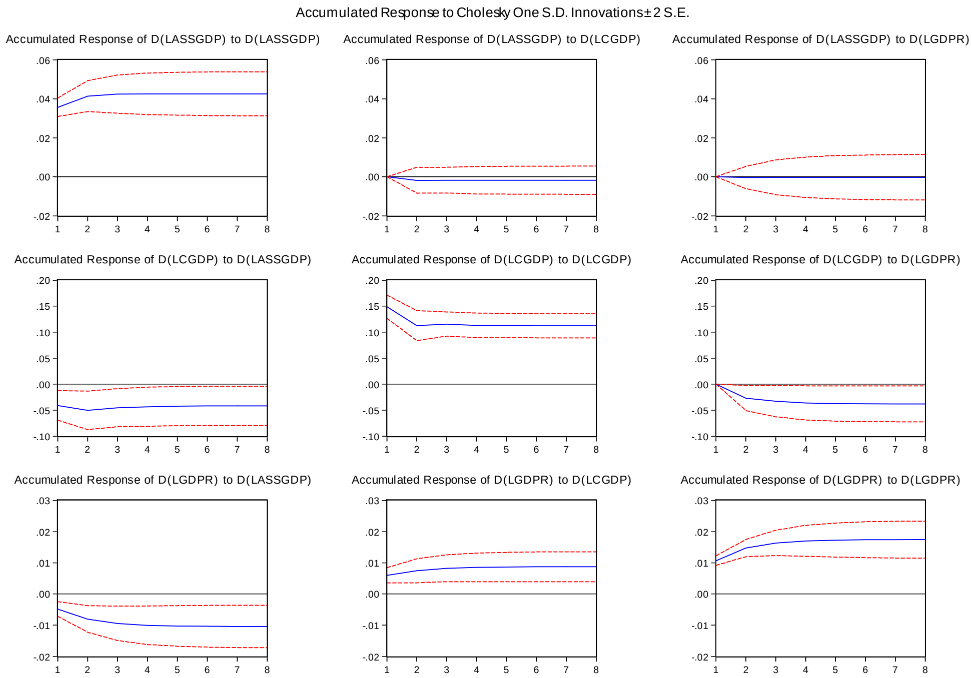



Source: own calculations.

Estimated values of response functions indicate that system is stable, after impulse shocks all variables tend to their normal values after about 4-6 quarters (see: Fig 2) and accumulated effect of shocks stabilise after 3-4 quarters (see: Fig 3).

Innovations in the growth rate of the ratio of MSF's assets to GDP significantly positively influence changes in its own growth rates and negatively the growth rate of the ratio of stock exchange market capitalisation to GDP and in growth rate of GDP. Unit shock in d(LCGDP) has almost no impact on d(LASSGDP) but support hypothesis about positive influence on growth rates. In case of accumulated effect results are similar, also effect of shock in changes in the ratio of assets is significant but negative, but ratio of stock market capitalisation has strong positive impact on growth rates.

**Fig 3. Accumulated response functions.**

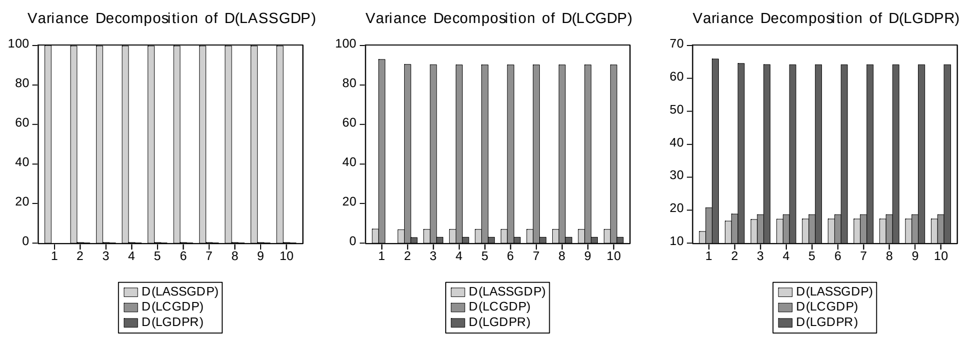



Source: own calculations.

Next, we can estimate the forecast error variance decomposition. The forecast error of $j-th$ variable can consist of all the innovations (although some can be zero) and the proportion of the forecast error variance of variable $j$ accounted for shock in $k-th$ variable can be interpreted as the contribution of innovations in variable $k$ to the forecast error variance of variable $j$. Forecast error variance decomposition is presented on Fig 4.

Almost 100% of forecast error variance of the ratio of assets to GDP is accounted for by own innovations, about 0,25% results from innovations in the ratio of capitalisation to GDP and only 0,01% form GDP. Similarly, the most of forecast error variance of d(LCGDP) and d(LGDP) is accounted for by their own innovations, in case of the stock exchange capitalisation 7% results from innovations in d(LASSGDP) and about 3% from innovations in GDP growth rate. About 17% of forecast error variance of GDP growth rate is accounted for by innovations in growth of MFI’s assets ratio and 19% by innovations in the stock exchange capitalisation.

**Fig 4. Forecast error variance decomposition.**




Source: own calculations.

## Conclusions

Results of the research are not straightforward, we can conclude that there are Granger casual relations between V4 countries. Granger causality tests proof existence of short-time causality between financial development and growth, but impulse response functions show that development of financial institutions has negative impact on GDP growth rates, but developing shares and bond market can lead to increase in economies’ growth. In further research the emphasis should be put on long-time relations (cointegration analysis should be included) and letting for non-linear dependencies (e.g. Hung [2009]).

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# INFORMATION AND COMMUNICATION TECHNOLOGY, REGIONAL CONVERGENCE AND GROWTH: AN EMPIRICAL EXAMINATION FROM A PANEL DATA[1]

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**Abstract:** This paper empirically explores the role of ICT in fostering economic growth and convergence in Europe. Based on a panel data of selected European economies including Slovakia during the period 1995-2008 and 2004-2008 seem to suggest that ICT helps to boost growth and reduce regional disparity. However, the results of this paper and previous other studies also indicate that ICT investment should be accompanied by other complementary investments to sustain growth and convergence.

**Keywords:** Information and communication technology, economic growth, regional convergence, panel data

## Introduction

The role information and communication technology (hence forth, ICT) plays not only in boosting productivity and economic growth but also in helping to reduce regional disparity have now been fully recognized by governments and businesses all around the world. Despite the current financial and economic turbulences that spark a downward spiral of low growth and high unemployment, the world economy, on average, has been

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performing extraordinarily well., mainly since the mid 1990s With the exception of short period downturns (1997-98 Asian financial crisis, a slowdown in Japan and the post 9/11 period in the US), the majority of countries in the world managed to achieve high rate of economic growth. The global growth was relatively higher since the mid 1990s as opposed to the late 1980s and early 1990s. For illustration, in 2007, of the 160 countries monitored by the United Nations, 102 had a growth rate of real GDP per capita of above 3% (UN, 2008). Likewise, unlike the early 1990s, both developed and emerging economies have experienced a significant degree of productivity convergence.

While there are several explanations behind the favourable dynamics of cross-country and cross-regional convergence since the mid 1990s, the expansion of information and communication technology is considered to be on the top of all. The results of this and a bulk of previous other empirical studies indicate that countries that have invested into information and communication technology (hence forth, ICT) and human capital have managed to achieve accelerated productivity performance and economic growth and lower subsequent regional disparities. In this respect, OECD (2003) study indicates that the contribution of ICT to OECD economies was significantly visible both in productivity performance and economic growth. "ICT investment in OECD countries rose from less than 15% of total non-residential investment in the early 1980s, to between 15% and 30% in 2001. Estimates show that it typically accounted for between 0.3 and 0.8 percentage points of growth in GDP and labour productivity over the 1995-2001 periods" (OECD, 2003). ICT's average contribution to French GDP growth was estimated to be approximately 0.2% per year over 1969 and 1999. This figure increased to 0.3% between 1995 and 1999 (Gilbert Cette, et al. 2001). Likewise, ICT contribution to US labor productivity was critical to its competitiveness. Kevin J. Stiroh (2002, p. 1560) indicate that „When 1995-2000 is compared to 1987-95, 26 IT-using industries contributed 0.83 percentage point to the aggregate productivity acceleration and the two IT-producing industries contributed 0.17. The remaining 33 industries made a negative contribution of 0.21 on net, suggesting that IT-related industries are indeed driving the U.S. productivity revival. Nonetheless, several studies also indicate number of outstanding issues in terms of ICT's contributions. First, empirical studies seem to suggest that investment into ICT does not guarantee a long term growth if there is a lack of other determinants, including but not limited to expenditures on research and development, human capital accumulation and other ICT infrastructures. Second, the existing significant cross-country variations in ICT investment continue to make cross-country variations in productivity and growth.

This paper is aimed at empirically investigating the causality not only between economic growth and convergence and ICT investment but also different determinants of ICT capital. The paper will also make a cross-country comparison as far as their position in ICT, human capital and innovation are concerned.

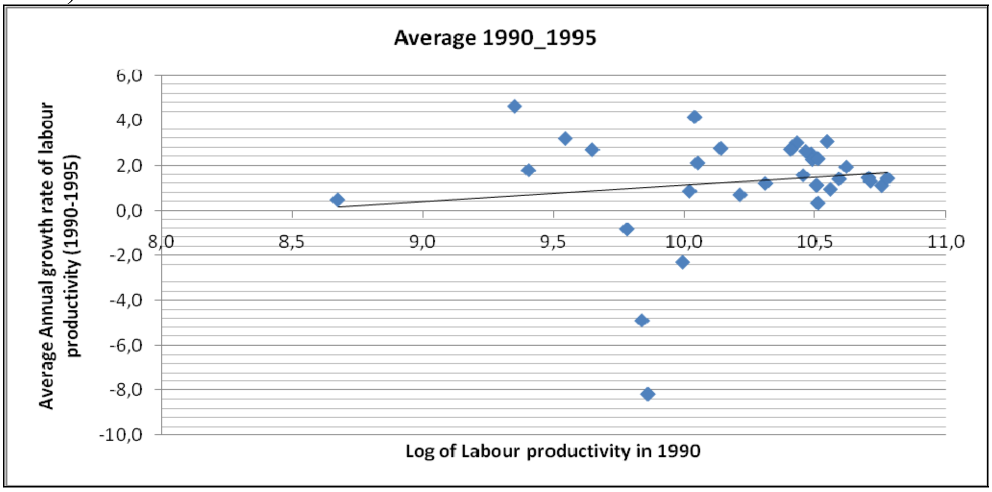
## 1. The causality between ICT investment and labor productivity

Following past studies, one of the most significant achievements of advanced and emerging economies in the past few years is considered to be the revival of labour productivity. Unlike the early 1990s where there was a greater degree of labour productivity divergence, the data from the mid 1990s seem to indicate a significant labour productivity convergence across countries (see graphs 1a and 1b). As was mentioned earlier, one of the reasons for this positive development is linked to the effective use and production of ICT in many of these countries (Ark, et al 2003; OECD, 2002, 2003; among others). The channels through which ICT affects labour productivity are numerous. First, ICT as an input increases the productivity of not only labour but also the non-IT capital. Second, through their network effects, ICT significantly reduce transaction costs for firms and hence help to improve overall efficiency in an economy. However, studies also indicate that most advanced European economies are lagging behind the US and other emerging economies' productivities primarily because of their lag in sufficient ICT investment and the heterogeneous policy environment that exists across European economies (EU ICT Task Force Report, 2006).

Table 1
Countries included in the productivity data

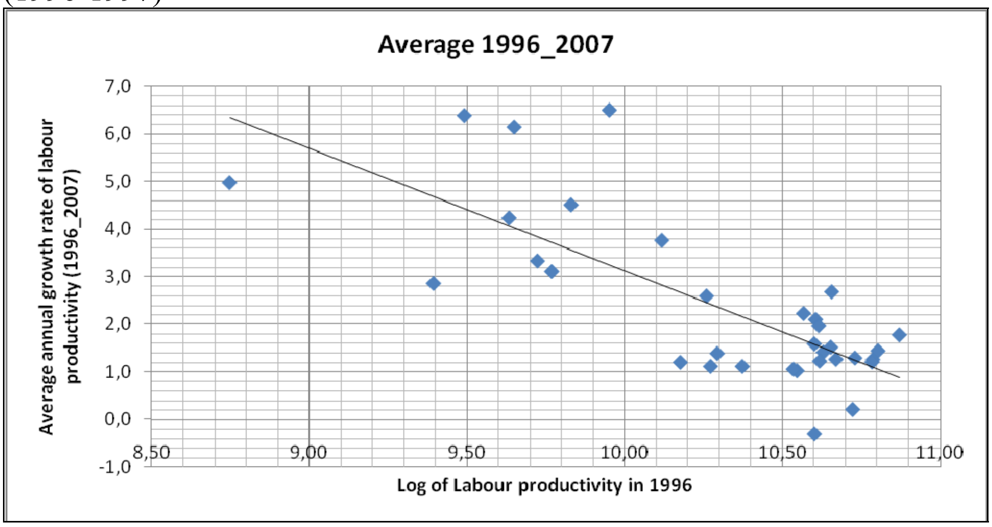
| | | | |
|---|---|---|---|
| Austria | Luxembourg | UK | Italy |
| Belgium | Malta | Canada | Hungary |
| Cyprus | Netherlands | United States | Latvia |
| Denmark | Norway | Australia | Lithuania |
| Finland | Portugal | New Zealand | Poland |
| France | Spain | Bulgaria | Romania |
| Germany | Sweden | Czech Republic | Slovak Republic |
| Greece | Switzerland | Estonia | Slovenia |
| Ireland | | | |

Graph 1a
Labor productivity divergence in advanced and emerging economies (1990-1995)




Source: own computation based on the Groningen data, 2009

Graph 1b
Labor productivity convergence in advanced and emerging economies (1996-1997)




Source: own computation based on the Groningen data, 2009

The regression results based on average GDP per capita growth on average level of labor productivity seem to suggest that the correlation between the two variables were the highest in in the late 1990s and 2000s periods compared with the 1970s and 1980s (see, table 3). The decade dummies both in the random effects and fixed effects models seem to be consistent with this conclusion (Buček and Workie, 2009).

Table 2
List of countries included in this study EU_14 (EU 15 without Germany) plus USA and Japan

| | |
|---|---|
| Austria | Italy |
| Belgium | Luxembourg |
| Denmark | Netherlands |
| Finland | Portugal |
| France | Spain |
| Greece | Sweden |
| Ireland | United Kingdom |

Table 3
Labor productivity and real GDP per capita: decade level averages (1960-2006[1]

| Variable | Cross-section | Random Effects | | Fixed Effects | |
|---|---|---|---|---|---|
| CONST | -1.69<br>(0.99) | -1.44***<br>(0.206) | 0.69<br>(0.88) | -1.43***<br>(0.206) | 1.59*<br>(0.933) |
| Log of labor productivity (level) | 1,07***<br>(0.154) | 1.04***<br>(0.016) | 0.85***<br>(0.072) | 1.04***<br>(0.017) | 0.77***<br>(0.082) |
| Dummy for 1970s | | | 0.053<br>(1.24) | | 0.089*<br>(0.046) |
| Dummy for 1980s | | | 0.094*<br>(0.057) | | 0.148**<br>(0.062) |
| Dummy for 1990s | | | 0.016**<br>(0.070) | | 0.233***<br>(0.078) |
| Dummy for 2000s | | | 0.314**<br>(0.123) | | 0.447***<br>(0.078) |
| No. of Groups | 16 | 16 | 16 | 16 | 16 |
| No. of Obs. | 16 | 80 | 80 | 80 | 80 |
| $R^2$ | 0.50 | 0.98 | 0.98 | 0.98 | 0.99 |

1. Dependnent variable_ Log of real GDP Per capita (Decade average: 1960-2006)

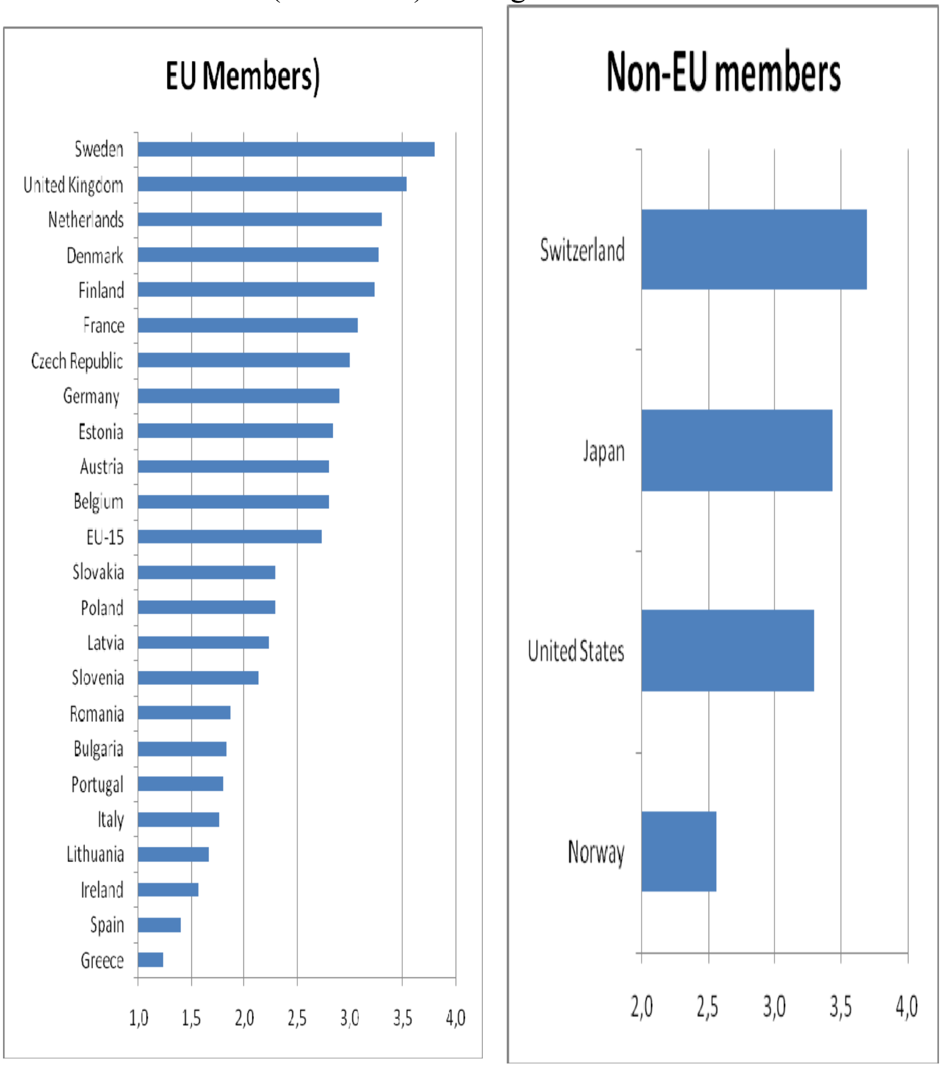
Source: Workie (2007)

## 2. Cross-country variation in ICT and ICT determinants

Despite the unambiguous contribution of informtion technology to labour productivity and economic growth, there is a visible cross-country variation when it comes to ICT investment. With the exception of the United Kingdom and scandinavian countries (Seden being on the top), other European countries are far behind United States and Japan (graph 2). This is indeed one of the major explanations behind Europe's lag in labour productivity revival in the context of the so-called digital era. The position of Slovakia in ICT investment is below the EU_15 average and far below the Czech Republic.

Similar results in table 4 indicates cross-country variation in household and government use of ICT services. From the table is clear that those economies that are behind in ICT use are on average lagging behind also in terms narrowing GDP per capita gap between them and the EU_15 average.

Graph 2
Investment into ICT (% of GDP) Average 2004-2006




Source: own computation based data from Eurostat, 2009.

Table 4
Cross-country variation in ICT use (household and government)_average 2006-2008

| | Internet_ Household[1] | E-government use_Firms[2] | E-government use_Individuals[3] |
|---|---|---|---|
| **Belgium** | 59 | 60 | 23 |
| **Bulgaria** | 20 | 50 | 7 |
| **Czech Republic** | 37 | 74 | 16 |
| **Denmark** | 80 | 88 | 48 |
| **Germany** | 71 | 54 | 36 |
| **Estonia** | 52 | 74 | 31 |
| **Ireland** | 57 | 88 | 28 |
| **Greece** | 26 | 81 | 10 |
| **Spain** | 45 | 60 | 27 |
| **France** | 51 | 69 | 37 |
| **Italy** | 43 | 84 | 16 |
| **Cyprus** | 40 | 54 | 16 |
| **Latvia** | 49 | 47 | 20 |
| **Lithuania** | 43 | 79 | 17 |
| **Luxembourg** | 75 | 86 | 49 |
| **Hungary** | 39 | 53 | 22 |
| **Malta** | 55 | 73 | 21 |
| **Netherlands** | 83 | 79 | 54 |
| **Austria** | 60 | 81 | 33 |
| **Poland** | 42 | 64 | 10 |
| **Portugal** | 40 | 69 | 18 |
| **Romania** | 22 | 40 | 6 |
| **Slovenia** | 57 | 82 | 30 |
| **Slovakia** | 44 | 83 | 29 |
| **Finland** | 69 | 94 | 50 |
| **Sweden** | 80 | 79 | 53 |
| **United Kingdom** | 67 | 57 | 35 |
| **Norway** | 77 | 74 | 60 |

1. Percentage of households having access to the Internet at home
2. Percentage of enterprises which use the Internet for interaction with public authorities
3. Percentage of individuals who have used the Internet, in the last 3 months, for interaction with public

Source“ own processing based on Eurostat 2009

## 3. The relationship between ICT and economic growth and convergence: A panel data analysis

I estimate the relationship between log of real GDP per capita growth and ICT, controlling for investments into non-ICT capital, intial GDP per capita (GDPI), education, (EDU) growth rate of population and vector of policy and other determinants of growth. The results are in table 5 and suggest that while the coefficients on ICT are positive in both the random and fixed effects models, they are statstically significant only in the later, indicating potentially missing variable problem in the random effects models. The coefficients on the intiial GDP per capita beinf negative and significant (in most cases) seem to suggest that there is conditional convergence across countries.

Table 5
The causality between and economic growth and convergence (1995-2008)[1]

| | Random Effects Model | | | Fixed Effects Model | | |
|---|---|---|---|---|---|---|
| Variables | Reg. 1 | Reg. 2 | Reg. 3 | Reg. 1 | Reg. 2 | Reg. 3 |
| Constant | 0.703<br>(1.3) | -0.019<br>(-0.04) | -0.079<br>(-0.14) | -3.29** <br>(-2.72) | -2.77** <br>(-2.39) | -2.14* <br>(-1.77) |
| Initial GDP per capita | -0.026* <br>(-1.83) | -0.038*** <br>(-2.8) | -0.041** <br>(-2.92) | -0.051* <br>(-1.96) | -0.039<br>(-1.55) | -0.04<br>(-1.62) |
| ICT | 0.001<br>(0.12) | 0.007<br>(1.00) | 0.05<br>(0.62) | 0.03** <br>(2.26) | 0.025** <br>(1.99) | 0.021* <br>(1.65) |
| Non_ICT | -0.024** <br>(-2.79) | -0.038*** <br>(-3.6) | -0.03*** <br>(-3.4) | -0.035* <br>(-1.99) | -0.037** <br>(-2.23) | -0.031* <br>(-1.81) |
| Life Exp. | -0.076<br>(-0.58) | 0.107<br>(0.78) | 0.122<br>(0.87) | 0.903** <br>(3.18) | 0.719** <br>(2.59) | 0.579** <br>(2.00) |
| Population growth | -0.014<br>(-0.03) | 0.014<br>(0.03) | -0.235<br>(-0.47) | -1.01** <br>(-2.02) | -0.96** <br>(-2.01) | -1.345** <br>(-2.49) |
| Inflation | -0.58*** <br>(-2.75) | -0.625*** <br>(-3.29) | -0.059** <br>-3.13) | -0.63*** <br>(-3.78) | -0.653*** <br>(-4.1) | -0.63*** <br>(-4.04) |
| Openess | | 0.015*** <br>(3.44) | 0.015*** <br>(3.3) | | 0.04** <br>(2.52) | 0.036** <br>(2.32) |

| | | | | | | |
|---|---|---|---|---|---|---|
| No. of Countries | 18 | 18 | 18 | 18 | 18 | 18 |
| No. of Observations | 70 | 70 | 70 | 70 | 70 | 70 |
| $R^2$ | 0.51 | 0.64 | 0.68 | 0.73 | 0.77 | 0.78 |
| Time Dummies | yes | Yes | yes | yes | Yes | yes |

1. Dependent variable is growth rate of real GDP per capita (panel: 1995-97; 1998-2000; 2001-2004; 2005-2008)

The empirical analysis based on annual pooled-cross setion time series for 23 European countries (listed in table 8) between the period 2004-2008 reaveal slightly different results (see, table 6). While non-ICT investments remain statistically insignificantly related to GDP per capita growth, in contrast to what we would expect, ICT investments remain statistically insignificant as well. However, this does not seem to suggest that ICT does not play a positive role in boosting economic growth, but rather it may well be due to the short time period (since effects of ICT investments come with time lag). Alternatively, the insignificant or even negative relationship between ICT and growth may be attributed to its low level. I therefore take the square of ICT investment (ICT_SQ), which changed the significance of the model, at least in the case of the random effects model. Moreover, the negative and statistically significant dummy on initial GDP per capita (GDP (t-1)) indicate that there is conditional convergence across countries controlling for ICT investments and other varables.

Table 6
The causality between ICT, growth and convergence (2004-2008)

| | Random Effects | | | Fixed Effects | | |
|---|---|---|---|---|---|---|
| VARIABLE | Reg1 | Reg2 | Reg3 | Reg1 | Reg2 | Reg3 |
| CONST | 0.87** (2.74) | 0.96** (2.98) | 0.997** (2.92) | 3.32*** (3.69) | 3.34*** (3.8) | 3.34*** (3.79) |
| GDP(t-1) | -0.049*** (-3.45) | -0.058** (-3.2) | -0.061*** (-3.25) | -0.288*** (-4.15) | -0.29*** (-4.27) | -0.332*** (-3.9) |
| ICT | -0.526** (-2.07) | -0.528** (-2.1) | -0.532** (-2.1) | -0.44 (-1.45) | -0.42 (-1.44) | -0.325 (-1.01) |
| ICT_SQ | 0.150** (2.14) | 0.149** (2.18) | 0.15** (2.15) | 0.121 (1.45) | 0.107 (1.31) | 0.083 (0.95) |
| Non-ICT | -0.0003 (-0.3) | 0.001 (0.95) | 0.001 (0.72) | 0.005 (1.2) | 0.004 (1.14) | 0.004 (1.21) |
| EDU | 0.029 (1.26) | 0.029 (1.25) | 0.027 (1.01) | -0.018 (-0.11) | -0.006 (-0.04) | -0.017 (-0.11) |
| CPI | | -0.007** (-2.4) | -0.007** (-2.4) | | -0.006** (-2.27) | -0.008** (-2.23) |
| No. of countries | 23 | 23 | 23 | 23 | 23 | 23 |
| No. obs | 92 | 92 | 92 | 92 | 92 | 92 |
| $R^2$ | 0.17 | 0.22 | 0.22 | 0.32 | 0.37 | 0.38 |

## 4. Determinants of ICT: Empirical investigation based on cross-section and correlation matrix

As has been mentioned in the first part, one of the main conclusion of theoretical and empirical studies on the contribution of ICT investments, for ICT to generate higher added value in respective economies, it must be accompanied by other complementary investments, such as education and research and development, more number of businesses and individulas connected through computer networks; e-government and other ingridients that increase the network effects of ICT caital. The correlation matrices in table 9 show that most these indicators are indeed corelated to each other.

Table 7
List of countries included in the study

| | |
|---|---|
| Austria | Portugal |
| Belgium | Spain |
| Denmark | Switzerland |
| Finland | UK |
| France | Bulgaria |
| Germany | Croatia |
| Greece | Czech Republic |
| Ireland | Hungary |
| Italy | Poland |
| Luxembourg | Romania |
| Netherlands | Slovakia |
| Norway | Slovenia |

Table 8
Selected Determinants of ICT based a cross-section data (2004-2007)

| Variable | Definition | Source |
|---|---|---|
| INTER | Percentage of households with access to the Internet at home | EUROSTAT |
| E_GOV Individual | Percentage of individuals who have used the Internet, in the last 3 months, for interaction with public | EUROSTAT |
| E_GOV Enterprises | Percentage of enterprises which use the Internet for interaction with public authorities | EUROSTAT |
| EDU | School attainment (Total) | EUROSTAT |
| LINE | Fixed line and mobile phone subscribers (per 1,000 people) | WDI (2007) |
| SAV | Gross Savings (% of GDP) | WDI (2007) |
| ICT | Information and Communication expenditure (% of GDP) | EUROSTAT |
| RES | Researchers in R&D (per million people) | WDI (2007) |
| TEL | Telephone mainlines (per 1,000 people) | WDI (2007) |
| GDP | Real GDP per capita (PPP adjusted) | WDI (2007) |

Table 9
Correlation matrix from a cross-section of economies (2004-2007)*

| | INTER | E_Gov | EDU | LINE | SAVING | ICT | RES | TEL | E_GOVI | GDP |
|---|---|---|---|---|---|---|---|---|---|---|
| INTER | 1 | | | | | | | | | |
| E_Gov | 0.378* (0.00) | 1 | | | | | | | | |
| EDU | 0.499* (0.00) | 0.303* (0.00) | 1 | | | | | | | |
| LINE | 0.616* (0.00) | 0.400* (0.00) | 0.211 (0.06) | 1 | | | | | | |
| SAVING | 0.556* (0.00) | 0.477* (0.00) | 0.184 (0.11) | 0.508* (0.00) | 1 | | | | | |
| ICT | 0.563* (0.00) | 0.182 (0.15) | 0.327* (0.01) | 0.408* (0.00) | 0.252* (0.04) | 1 | | | | |
| RES | 0.747* (0.00) | 0.574* (0.00) | 0.530* (0.00) | 0.447* (0.00) | 0.467* (0.00) | 0.588* (0.00) | 1 | | | |
| TEL | 0.552 (0.00) | 0.189* (0.00) | 0.399* (0.00) | 0.676* (0.00) | 0.252* (0.02) | 0.384* (0.00) | 0.467* (0.00) | 1 | | |
| E_GOVI | 0.898* (0.00) | 0.427* (0.00) | 0.484* (0.00) | 0.525* (0.00) | 0.612* (0.00) | 0.543* (0.00) | 0.818* (0.00) | 0.502* (0.00) | 1 | |
| GDP | 0.733* (0.00) | 0.386* (0.00) | 0.546* (0.00) | 0.716* (0.00) | 0.499* (0.00) | 0.447* (0.00) | 0.569* (0.00) | 0.786* (0.00) | 0.664* (0.00) | 1 |

*The asterik indicate significance at 5% level.

## 5. Conclusion and Policy Implications

The results of this paper and a bulk of other studies indicate that ICT investment plays a pivotal role in accelerating productivity and economic growth and ultimately helping countries with low income per capita to catch up with those in the opposite camp. It is also critical that policy makers understand thatg global competition is going to be determined by competitive advantage rather than by comparative advantage as was the case in the past. In this regard, innovation will remain the sole source of sustaining competitiveness. Those lagging behind in R&D and ICT investments are condemned to lag in everything else, including regional disparity. High quality and globally competent education system will remain at the heart of innovation and efficiency. As *Charles* Darw*in* pointed out long time ago**,** *"It's not the strongest nor most intelligent of the species that survive; it is the one most adaptable to change".* Therefore, in order to reduce regional disparity and achieve accelaretd growth, it is critically necessary to investment into ICT, education, research and development.

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# DESCRIBING WORLD FINANCIAL DEVELOPMENT WITH THE USE OF DYNAMIC MULTIDIMENSIONAL COMPARATIVE ANALYSIS

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## Introduction

The role of financial markets and their development is strongly emphasized in recent studies on business cycles and economic growth. But there is not much empirical research on financial sector's development in macro scale. The main reason is the lack of sufficient data, such as households' survey data, quasi-financial institutions' data or because of not having cross-countries homogenous methodology. So the key is the database of financial development indicators which would be comparable among countries. Such database have been constructed and updated by Thorsten Beck, Asli Demirgüç-Kunt and Ross Levine using International Financial Statistics (IFS) time series (See more in: Beck, Demirgüç-Kunt, Levine, 2000). It includes yearly indicators of all countries of the world from 1960, though some countries have incomplete data. Levine et al. suggested measuring method of financial development relative to GDP of the particular country.

## 1. Financial development theories

In general, modern theories of economic growth indicate two channels by which financial sector influences long-term growth:

- stimulating accumulation of the capital (not only physical, but also human capital),
- influence on rate of growth of technical progress.

Levine (1997) has selected five fundamental functions of financial intermediation institutions which positively affect economic growth. These are savings mobilization, risk management, gaining information about investing opportunities, monitoring and corporate controlling, facilitating of an exchange of goods and services.

Causality analysis of financial sector development and long-term growth was initiated by Walter Bagehot and Joseph Schumpeter. According to their theories, financial services are one of the most important catalysts of economic growth.

There is a lot of research which confirm a strong positive relationship between development of financial sector and economic growth (i.e. Goldsmith 1969; Roubini and Sala-I-Martin 1992; King and Levine 1993; Easterly 1993; Levine, Loayza and Beck 2000; Eber 2000; Lubecki 2004; Trabelsi 2004). Gurley and Shaw (1967) as well as Jung (1986) claimed, that highly probable is reverse direction. It means that economic growth generates increased demand for financial services which cause expansion of financial sector. Patrick (1966) suggested the hypothesis of stage of development. It states that on early stages of economic development the financial sector development leads to economic growth, but this influence gradually weakens as long as the economy grows. After crossing specific threshold the influence of economic growth on financial sector development begins to dominate.

Recent research (i.e. Calderon and Liu, 2003) show that there is a feedback between financial development and economic growth although financial development is more significant in case of developing countries and in the long-run.

## 2. Measuring the world financial development

This research uses mentioned above financial development database created and updated by Levine et al. The data until the year 2007 were used, so time series end just before world financial crisis. All indicators selected to the multidimensional analysis are relative to GDP. Their increase means that financial variable goes up more than GDP. There are, of course, some drawbacks of such approach. Its effect on the results is diminished by very careful interpretations as well as by considering the long-run patterns of time series (long and smoothed time series). The latter one must be done because of possible different time-shifts of the business cycle in the nominator and denominator. The sample consists of time series of 40 countries from 1990 to 2007. The list of selected countries as well as their group of income per capita (according to the World Bank) is presented in Table 1.

In order to synthesize information included in the database, taxonometric tools have been used – ordering based on dynamic synthetic variables as well as classification procedures.

The first very important step of the comparative analysis is the selection of variables. There are in general two selection's criteria – logical and statistic (empirical). Among 22 indicators describing financial sector in Levine's database, 9 indicators have been chosen, with respect to mentioned criteria[1]. More detailed description of selected indicators can be found in Table 2. Dynamic taxonomic measures of development are important part of dynamic comparative multidimensional analysis (DCMA).[2]

[1] Statistic verification was made with the use of correlation threshold. The important assumption of such analysis is that variables are not strongly correlated.
[2] DCMA was initiated in Poland by M. Cieślak (See: Cieślak 1976)

Let assume that the analysis is made for $T$ time units. Gathered information (data) can be presented in the block matrix of observations

$$X = \left[ X^1 \quad X^2 \quad \dots \quad X^T \right] \tag{1}$$

where $\mathbf{X^t}$ ($t=1,2,\dots,T$) is the matrix of observations in the year $t$:

$$X^t = \begin{bmatrix} x_{11}^t & x_{12}^t & \dots & x_{1K}^t \\ x_{21}^t & x_{22}^t & \dots & x_{2K}^t \\ . & . & . & . \\ x_{N1}^t & x_{N2}^t & \dots & x_{NK}^t \end{bmatrix} \tag{2}$$

In this matrix $x_{ik}^t$ ($k=1,2,\dots,K$, $i=1,2,\dots,N$, $t=1,2,\dots,T$) is the value of the diagnostic characteristic of $X_k$ in the object $O_i$ in the year $t$.

Dynamic taxonomic measure of development is the function (Nowak 1990, p. 162) $Z = f(X_1, X_2, \dots, X_K)$ which transforms three-dimensional matrix of observations **X**, described by the formula (1) into **Z**-matrix of the size $[N \times T]$:

$$Z = \begin{bmatrix} z_{11} & z_{12} & \dots & z_{1T} \\ z_{21} & z_{22} & \dots & z_{2T} \\ . & . & . & . \\ z_{N1} & z_{N2} & \dots & z_{NT} \end{bmatrix}. \tag{3}$$

In matrix (3) $z_{it}$ is the taxonomic measure of development of the $O_i$ object in $t$-time unit. In order to compute dynamic taxonomic measure of development, firstly all diagnostic variables ought to be normalized. In such dynamic approach the general normalization formula can be written as:

$$z_{ik}^t = \frac{x_{ik}^t}{x_{0k}} \tag{4}$$

where $z_{ik}^t$ is normalized value of *k*- diagnostic characteristic of *i*- object in *t*-time unit, and $x_{0k}$ is the base (denominator) of normalization of *k*-characteristic.[3] In this research we use following formula:

$$x_{0k} = \frac{1}{TN}\sum_{t=1}^{T}\sum_{i=1}^{N} x_{ik}^t \ . \tag{5}$$

Finally, values of synthetic taxonomic measure are derived from:

$$z_{it} = \frac{1}{K}\sum_{k=1}^{K} z_{ik}^t \ . \tag{6}$$

High values of this measure mean that the given object is more developed (has financial sector more developed). If $z_{it} > 1$, the object has higher level of development than average level in the whole set of objects in all cross-time database. If $z_{it} < 1$, the object has reached lower level of development comparing to whole set of objects in all cross-time database.

The next step of analysis is estimation of dynamics of relative financial development – average increase and average rate of growth (Nowak 1990, p. 163).

The average increase measure of synthetic indicator of *i*-object is $b_i$ – parameter of linear trend: $\hat{Z}_i = a_i + b_i t$ ($t$=1,2,…,$T$), whereas the measure of average rate of growth is the value of $c_i = C_i - 1$ where $C_i$ is the parameter of exponential trend: $\hat{Z}_i = d_i C_i^{\,t}$.

After deriving such dynamic characteristics of synthetic variables, all objects can be classified using selected procedure. The base is the matrix of distances between $b_i$s or $c_i$s:

[3] More about possible bases of normalization for dynamic taxonometric analysis is presented i.e. in Nowak (1990), p. 163

$$\mathbf{D} = \begin{bmatrix} 0 & d_{12} & \ldots & d_{1N} \\ d_{21} & 0 & \ldots & d_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ d_{N1} & d_{N2} & \ldots & 0 \end{bmatrix} \quad \text{where } d_{ij} = \frac{c_i - c_j}{\max(c_i, c_j)} \quad (d_{ij} \in <0,1>).$$

In this research, Czekanowski's classification method and PAM (Partioning Around Medoids) have been used in order to identify typological groups of countries of the similar level of dynamics (dynamics of financial development).[4] The results obtained from both procedures are almost the same (See more about classification methods in Pluta 1977; Grabiński, Wydymus, Zeliaś 1989; Nowak 1990; Kolenda 2006).

CzeKo classification algorithm is based on **D**-matrix where elements near main diagonal are possibly the lowest ones, and they rise as they are more far from the main diagonal. The tool which facilitates such ordering is matrix of weights - $\mathbf{U}[u_{ij}]$ for *i,j=1,2,…,n*. Matrix **U** is the pattern for **D**-matrix. Desired sequence in **D**–matrix can be achieved by maximizing following function:

$$F = \sum_{i=1}^{n} \sum_{j>i}^{n} d_{ij} u_{ij} \to \max. \tag{7}$$

and testing it's value for different orders. Weight matrix can be derived from following formula:

$$u_{i,j} = \frac{1}{n(n-1)} \left[ 2n|i-j| + 2 - i - j - (i-j)^2 \right]. \tag{8}$$

PAM method (Partioning Around Medoids) first computes *k* representative objects, called *medoids.* A medoid can be defined as that object of a cluster, whose average dissimilarity to all the objects in the cluster is minimal. After

[4] Czekanowski's method applied in this research was slightly modified. Instead of colouring classical Czekanowski's diagram the CzeKo algorithm was used – see Kolenda (2006) p. 80-84.

finding the set of medoids, each object of the data set is assigned to the nearest medoid. The key role plays the distance from the medoid. It should be the shortest. The second step is verification and possible change of the set of representatives. For the new set of medoids again clustering is carried out. It must be checked whether after exchanging *l*-representative ($r_l$) for *j*-object the following function's value has decreased:

$$F = \frac{\sum_{i=1}^{n} d(i^{(l)}, r_l)}{n}$$, where $l$ – the group number for *i*-object, $r_l$ – *l*-representative.

If value of $F$ decrease, then given $r_l$– representative is exchanged for *j*-object.

## 3. Empirical results

It must be emphasized, that computed dynamic synthetic indicators, though comparison able cross-time and cross-countries, their values are dimensionless, artificial, so it is difficult to interpret single indicator economically.

Values of dynamic synthetic indicators included in Table 3 suggest, that almost all concerned countries have experienced year-to-year relative development of financial sector. Comparing the levels of synthetic indicators they can be easily divided into two groups:

a) of lower development (values of indicators below 0,80), with such countries like: Peru, Mexico, Columbia, Argentina, Poland, Indonesia, Turkey, Slovakia, India, Hungary, Brazil, Philippines, Czech Republic, Chile (Picture 1),

b) of higher development (values of indicators exceeding 1,20), including: Switzerland, USA, Japan, UK, Netherlands.

Another interesting remark is that there is visible cross-time synchronicity among European countries, especially within the group: Belgium, France, Germany, Italy, Portugal, Spain, Sweden. Surprisingly high increase of indicator is noted in case of Island – from 0,57 in 1995 to 2,64 in 2005. It is an extraordinary change over all sample of countries. The reason of such unusual growth was 8-times increase in capitalization of private sector's bond market in relation to GDP (prbond) and over 15-times raise of stock market capitalization to GDP (with even more increase in turnover) and more than 4-times growth of loans for private sector and commercial banks' assets in relation to GDP. Natural conclusion is that dynamics (average increase and average rate of growth) of indicator in case of Island are the highest ones (See Table 4). Next to Island, the highest rate of relative development had Turkey, Indonesia, Spain, Argentina, Poland, Greece, Irland, Denmark, Finland (more than 5% growth year-to-year).[5] Classification of countries (Czekanowski's and PAM procedures assuming 13 clusters) according to the synthetic indicator's rate of growth gave following results – see Table 5. Poland is in the same group as Spain or Ireland, and Slovakia in the same cluster with Germany. Results don't prove that emerging/developing countries have higher rate of relative financial growth than high income countries.

[5] Stability of parameter's estimates of trend functions was tested in first step by using recursive coefficient estimates procedure in order to select structural breaks, and in second step by using the Chow breakpoint test. Only half of estimated equations have stable parameters. Most significant structural breaks were after the year 2000.

**Table 1**
Countries selected to the research and their income group (according to the World Bank)

| Country | abbrev. | income group | Country | abbrev. | income group |
|---|---|---|---|---|---|
| Argentina | ARG | High income | Island | ISL | High income |
| Australia | AUS | High income | Japan | JPN | High income |
| Austria | AUT | High income | Korea South | KOR | High income |
| Belgium | BEL | High income | Malesia | MYS | Upper middle income |
| Brazylia | BRA | Lower middle income | Mexico | MEX | Upper middle income |
| Canada | CAN | High income | Netherlands | NLD | High income |
| Chile | CHL | Upper middle income | Norway | NOR | High income |
| Columbia | COL | Lower middle income | Peru | PER | Lower middle income |
| Czech Rep. | CZE | Upper middle income | Poland | POL | Upper middle income |
| Denmark | DNK | High income | Portugal | PRT | High income |
| Filipines | PHL | Lower middle income | Rep. of South Africa | ZAF | Upper middle income |
| Finland | FIN | High income | Singapur | SGP | High income |
| France | FRA | High income | Slovakia | SVK | Upper middle income |
| Germany | DEU | High income | Spain | ESP | High income |
| Greece | GRC | High income | Sweden | SWE | High income |
| Hungary | HUN | Upper middle income | Switzerland | CHE | High income |
| India | IND | Low income | Thailand | THA | Lower middle income |
| Indonesia | IDN | Lower middle income | Turkey | TUR | Upper middle income |
| Ireland | IRL | High income | United Kingdom | GBR | High income |
| Italy | ITA | High income | USA | USA | High income |

**Table 2**
Names and descriptions of diagnostic variables (features) selected to the research from IFS (Levine et al.) database. They all represent development of financial sector in selected countries (objects).

| abbrev. | name | description |
|---|---|---|
| dbagdp | Deposit Money Bank Assets / GDP | Claims on domestic real nonfinancial sector by deposit money banks as a share of GDP |
| pcrdbofgdp | Private Credit / GDP | Private credit by deposit money banks and other financial institutions to GDP |
| fdgdp | Financial System Deposits / GDP | Demand, time and saving deposits in deposit money banks and other financial institutions as a share of GDP, |
| stmktcap | Stock Market Capitalization / GDP | Value of listed shares to GDP |
| stvaltraded | Stock Market Total Value Traded / GDP | Total shares traded on the stock market exchange to GDP |
| prbond | Private Bond Market Capitalization / GDP | Private domestic debt securities issued by financial institutions and corporations as a share of GDP |

| | | |
|---|---|---|
| pubond | Public Bond Market Capitalization / GDP | Public domestic debt securities issued by government as a share of GDP |
| inslife | Life Insurance Premium Volume / GDP | Life Insurance Premium Volume to GDP |
| insnonlife | Non-Life Insurance Premium Volume / GDP | Non-Life Insurance Premium Volume to GDP |

**Remarks:**
All financial variables in nominator were denominated (by authors of database) using CPI (*Consumer Price Index*) of particular country. Additionally Authors smoothed time series of financial variables using simple average (k=2). Denominator of each listed above diagnostic indicator is in real terms. (See more in Beck, Demirgüç-Kunt, Levine, 2000)

**Picture 1**
Values of dynamic synthetic indicators. The group of countries with the lowest level of financial sector development relative to GDP.

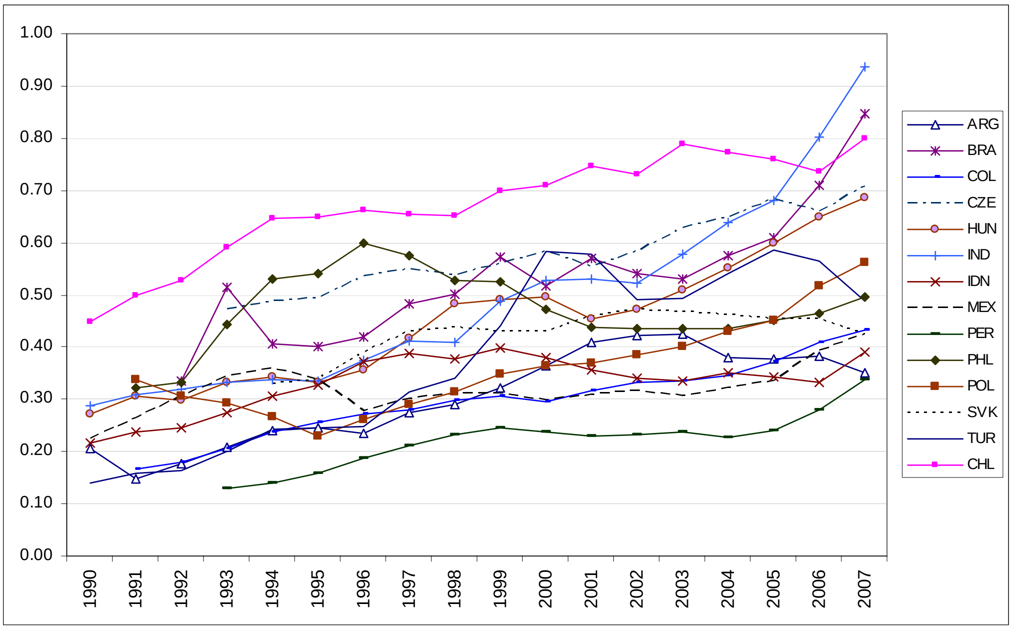

**Table 3** Values of dynamic synthetic indicators in selected years.

| 1990 | | 1993 | | 1995 | | 1998 | | 2000 | | 2003 | | 2005 | | 2007 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| JPN | 1.707 | MYS | 1.708 | NLD | 1.914 | ZAF | 1.470 | CHE | 2.566 | CHE | 2.208 | ZAF | 1.592 | ISL | 3.285 |
| CHE | 1.402 | JPN | 1.679 | CHE | 1.779 | USA | 1.901 | USA | 2.410 | USA | 1.980 | USA | 2.115 | CHE | 2.922 |
| USA | 1.300 | CHE | 1.638 | JPN | 1.744 | TUR | 0.340 | NLD | 2.006 | GBR | 1.785 | TUR | 0.586 | GBR | 2.666 |
| GBR | 1.175 | USA | 1.443 | USA | 1.525 | THA | 0.822 | GBR | 1.966 | JPN | 1.761 | THA | 0.929 | USA | 2.513 |
| ZAF | 1.172 | GBR | 1.289 | MYS | 1.449 | SWE | 1.231 | JPN | 1.668 | NLD | 1.734 | SWE | 1.493 | NLD | 2.294 |
| NLD | 1.062 | SGP | 1.274 | ZAF | 1.343 | SVK | 0.439 | FIN | 1.659 | DNK | 1.633 | SVK | 0.453 | JPN | 2.231 |
| DNK | 1.055 | BEL | 1.249 | GBR | 1.330 | SGP | 1.196 | KOR | 1.658 | SGP | 1.596 | SGP | 1.557 | DNK | 2.032 |
| BEL | 0.980 | NLD | 1.165 | BEL | 1.260 | PRT | 1.016 | ZAF | 1.581 | ISL | 1.488 | PRT | 1.153 | ZAF | 1.985 |
| KOR | 0.960 | ZAF | 1.155 | SGP | 1.180 | POL | 0.314 | SWE | 1.473 | ZAF | 1.480 | POL | 0.451 | ESP | 1.981 |
| SWE | 0.921 | DNK | 1.127 | FRA | 1.094 | PHL | 0.528 | BEL | 1.460 | KOR | 1.452 | PHL | 0.450 | IRL | 1.953 |
| FRA | 0.921 | SWE | 1.075 | DNK | 1.088 | PER | 0.233 | DNK | 1.440 | BEL | 1.426 | PER | 0.241 | KOR | 1.909 |
| CAN | 0.879 | FRA | 1.075 | CAN | 1.067 | NOR | 0.804 | ESP | 1.411 | MYS | 1.423 | NOR | 0.963 | SGP | 1.896 |
| MYS | 0.873 | KOR | 1.066 | KOR | 1.054 | NLD | 2.138 | DEU | 1.406 | IRL | 1.326 | NLD | 1.997 | CAN | 1.788 |
| AUT | 0.852 | CAN | 1.034 | SWE | 1.051 | MYS | 1.376 | FRA | 1.402 | FRA | 1.302 | MYS | 1.391 | SWE | 1.763 |
| NOR | 0.780 | DEU | 0.967 | DEU | 1.042 | MEX | 0.312 | SGP | 1.399 | ESP | 1.291 | MEX | 0.334 | FRA | 1.674 |
| AUS | 0.770 | FIN | 0.896 | ITA | 0.935 | KOR | 1.249 | MYS | 1.393 | DEU | 1.283 | KOR | 1.505 | AUS | 1.651 |
| FIN | 0.761 | ITA | 0.896 | AUS | 0.927 | JPN | 1.673 | IRL | 1.297 | SWE | 1.269 | JPN | 2.005 | FIN | 1.549 |
| ITA | 0.754 | AUT | 0.884 | AUT | 0.922 | ITA | 1.050 | CAN | 1.291 | ITA | 1.251 | ITA | 1.366 | PRT | 1.515 |
| ESP | 0.685 | AUS | 0.859 | IRL | 0.865 | ISL | 0.694 | ITA | 1.269 | AUS | 1.206 | ISL | 2.491 | BEL | 1.506 |
| PRT | 0.596 | THA | 0.788 | FIN | 0.844 | IRL | 1.138 | AUS | 1.212 | CAN | 1.201 | IRL | 1.490 | ITA | 1.488 |
| THA | 0.511 | ESP | 0.777 | ESP | 0.805 | IND | 0.410 | PRT | 1.178 | PRT | 1.200 | IND | 0.681 | MYS | 1.473 |
| GRC | 0.485 | PRT | 0.723 | PRT | 0.790 | IDN | 0.378 | GRC | 1.045 | FIN | 1.194 | IDN | 0.344 | DEU | 1.384 |
| CHL | 0.449 | NOR | 0.712 | THA | 0.779 | HUN | 0.482 | AUT | 0.984 | AUT | 1.004 | HUN | 0.598 | AUT | 1.169 |
| IND | 0.287 | CHL | 0.590 | NOR | 0.715 | GRC | 0.735 | ISL | 0.958 | THA | 0.956 | GRC | 0.982 | NOR | 1.119 |
| HUN | 0.272 | GRC | 0.525 | CHL | 0.650 | GBR | 1.598 | THA | 0.781 | GRC | 0.871 | GBR | 2.012 | GRC | 0.945 |
| MEX | 0.224 | BRA | 0.514 | ISL | 0.557 | FRA | 1.169 | NOR | 0.756 | NOR | 0.844 | FRA | 1.447 | IND | 0.936 |
| IDN | 0.218 | CZE | 0.471 | GRC | 0.548 | FIN | 1.008 | CHL | 0.710 | CHL | 0.788 | FIN | 1.344 | THA | 0.934 |
| ARG | 0.205 | PHL | 0.443 | PHL | 0.541 | ESP | 1.097 | TUR | 0.584 | CZE | 0.627 | ESP | 1.539 | BRA | 0.846 |
| TUR | 0.140 | MEX | 0.342 | CZE | 0.492 | DNK | 1.274 | CZE | 0.584 | IND | 0.578 | DNK | 1.821 | CHL | 0.801 |
| | | HUN | 0.332 | BRA | 0.402 | DEU | 1.220 | IND | 0.528 | BRA | 0.531 | DEU | 1.305 | CZE | 0.708 |
| | | IND | 0.332 | MEX | 0.337 | CZE | 0.537 | BRA | 0.518 | HUN | 0.509 | CZE | 0.683 | HUN | 0.687 |
| | | POL | 0.293 | SVK | 0.337 | COL | 0.298 | HUN | 0.495 | TUR | 0.494 | COL | 0.373 | POL | 0.562 |
| | | IDN | 0.273 | IND | 0.336 | CHL | 0.653 | PHL | 0.471 | SVK | 0.468 | CHL | 0.759 | PHL | 0.497 |
| | | ARG | 0.208 | HUN | 0.333 | CHE | 2.469 | SVK | 0.431 | PHL | 0.434 | CHE | 2.403 | TUR | 0.487 |
| | | COL | 0.207 | IDN | 0.328 | CAN | 1.200 | IDN | 0.380 | ARG | 0.426 | CAN | 1.309 | COL | 0.433 |
| | | TUR | 0.202 | COL | 0.256 | BRA | 0.501 | POL | 0.365 | POL | 0.401 | BRA | 0.610 | MEX | 0.426 |
| | | PER | 0.128 | TUR | 0.245 | BEL | 1.434 | ARG | 0.365 | COL | 0.335 | BEL | 1.667 | SVK | 0.425 |
| | | | | ARG | 0.244 | AUT | 0.960 | MEX | 0.299 | IDN | 0.334 | AUT | 1.082 | IDN | 0.389 |
| | | | | POL | 0.230 | AUS | 1.103 | COL | 0.296 | MEX | 0.307 | AUS | 1.312 | ARG | 0.350 |
| | | | | PER | 0.159 | ARG | 0.289 | PER | 0.237 | PER | 0.237 | ARG | 0.377 | PER | 0.339 |

**Table 4**
Estimations of average increase ($b_i$) and rate of growth ($c_i$) of countries' synthetic variables from 1994 to 2007 basing on linear and exponential trend function.

| | $b_i$ | $c_i$ | | $b_i$ | $c_i$ | | $b_i$ | $c_i$ | | $b_i$ | $c_i$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ISL | 0.213 | 0.163 | GBR | 0.08 | 0.045 | SGP | 0.048 | 0.034 | JPN | 0.034 | 0.018 |
| IND | 0.04 | 0.076 | GRC | 0.032 | 0.044 | SWE | 0.045 | 0.034 | THA | 0.015 | 0.017 |
| TUR | 0.027 | 0.073 | FIN | 0.049 | 0.044 | CAN | 0.042 | 0.032 | CHL | 0.012 | 0.017 |
| ESP | 0.077 | 0.063 | PRT | 0.045 | 0.044 | NOR | 0.027 | 0.031 | AUT | 0.017 | 0.017 |
| POL | 0.023 | 0.063 | ARG | 0.013 | 0.044 | FRA | 0.04 | 0.03 | BEL | 0.021 | 0.015 |
| IRL | 0.074 | 0.06 | COL | 0.013 | 0.041 | CHE | 0.064 | 0.029 | MEX | 0.005 | 0.014 |
| HUN | 0.024 | 0.052 | KOR | 0.054 | 0.04 | CZE | 0.016 | 0.027 | NLD | 0.022 | 0.013 |
| DNK | 0.073 | 0.051 | ITA | 0.043 | 0.038 | ZAF | 0.039 | 0.026 | IDN | 0.001 | 0.002 |
| BRA | 0.025 | 0.046 | AUS | 0.044 | 0.037 | DEU | 0.024 | 0.02 | MYS | -0.018 | -0.012 |
| PER | 0.01 | 0.046 | USA | 0.066 | 0.035 | SVK | 0.008 | 0.02 | PHL | -0.01 | -0.019 |

**Table 5**
Results of classification made according to the criterion of rate of growth of dynamic synthetic indicators using Czekanowski's and PAM method.

| group number | cluster's elements using Czekanowski's method | cluster's elements using PAM method | group number | cluster's elements using Czekanowski's method | cluster's elements using PAM method |
|---|---|---|---|---|---|
| 1 | ISL | ISL | 8 | DEU, SVK | DEU, SVK |
| 2 | IND, TUR | IND, TUR | 9 | JPN, THA | JPN, THA, AUT, CHL |
| 3 | IRL, POL, ESP | IRL, POL, ESP | 10 | AUT, CHL, IDN | IDN |
| 4 | DNK, HUN | DNK, HUN | 11 | BEL | BEL, MEX, NLD |
| 5 | ARG, AUS, BRA, COL, FIN, GRC, ITA, KOR, PER, PRT, SGP, SWE, GBR, USA | ARG, BRA, COL, FIN, GRC, KOR, PER, PRT, GBR, | 12 | MEX, NLD | MYS |
| 6 | CAN, FRA, NOR, CHE | AUS, ITA, SGP, SWE, USA | 13 | MYS, PHL | PHL |
| 7 | CZE, ZAF | CAN, FRA, NOR, CHE, CZE, ZAF | | | |

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# A note on linear regression with interval data and linear programming

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**Abstract.** We study the set $B$ of OLS solutions of a linear regression model with interval-censored observations of the dependent variable. We show that the set $B$ is a zonotope. We describe it by a set of generators. We study the combinatorial complexity of $B$ and discuss show how the complexity is related to the design matrix of the regression model. We show that deciding, whether a given $\beta$ (i) is admissible, (ii) is extremal, (iii) is a vertex of $B$ are reducible to linear programming, and hence decidable in polynomial time. We show how to use this result in Monte Carlo estimation of the volume of $B$.

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**JEL classification:** C52.
**Keywords.** Interval data; interval regression; linear programming; zonotope.

## 1 Introduction

The linear regression model $\mathsf{E}(y) = X\beta$ describes the response of the dependent variable $y$ as a linear function of dependent variables $X$. The vector $\beta$ of regression parameters is unknown and it is to be estimated. The most common estimator, ordinary least squares (OLS), corresponds to finding $\widehat{\beta}$ such that the $L_2$-norm of residuals $y - X\widehat{\beta}$ is minimized.

The usual approach assumes that the observed values of dependent variables (the rows of the matrix $X$) and observations of the dependent variable (the components of the vector $y$) are *crisp*, i.e. they are real numbers. In many practical applications, some or all of the values $X$ and $y$ cannot be directly observed; they might be uncertain or fuzzy. Only an interval, in which the unobservable value is guaranteed to be, is known. In this context it is natural to generalize the linear regression model to be able to handle intervals.

Interval variables appear in economic and financial applications quite often. For example:

- traded variables have bid-ask spread;
- credit rating grades can be sometimes regarded as intervals of credit spreads above the risk-free yield curve;
- if we measure economic variables such as personal income, we sometimes obtain underestimated observations due to the presence of the 'grey zone'. If personal income $Y$ is measured by means of the income declared, the true income is likely to be in an interval $[Y, Y + \Delta_Y]$ where $\Delta_Y$ is an upper bound for 'grey' (undeclared) income;

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- there is a similar problem with demographic data, e.g. true immigration is in an interval $[I, I+\Delta_I]$ where $I$ is observed legal immigration and $\Delta_I$ is an (estimated) upper bound on illegal immigration;
- dynamic macroeconometric regression models often include variables such as foreign exchange rates or interest rates, that are not constant within a given period. Usually, the median or the average value is taken as a proxy. However, it might be more appropriate to regard that variable as an interval within which the variable changes over the period.

Further applications of interval data analysis in econometrics and finance are discussed in [4, 5, 6, 12].

Let $\underline{X} \leq \overline{X}$ be two $n \times p$ real matrices ("$\leq$" is understood componentwise). The *interval matrix* $\boldsymbol{X} := [\underline{X}, \overline{X}]$ is the set

$$\{X \in \mathbb{R}^{n\times p} : (\forall i \in \{1,\dots,n\})(\forall j \in \{1,\dots,p\})\ \underline{X}_{ij} \leq X_{ij} \leq \overline{X}_{ij}\},$$

where $X_{ij}$ denotes the $(i,j)$-th component of the matrix $X$. The *interval vector* $\boldsymbol{y} := [\underline{y}, \overline{y}]$ is a special case with one column. The interval vector $\boldsymbol{y}$ may be also regarded as an $n$-dimensional cube $\{y : \underline{y} \leq y \leq \overline{y}\}$.

In this text we deal with the *set of OLS-solutions* of an interval regression model. Throughout the text, $n$ shall stand for the number of observations and $p$ for dimension (=number of parameters).

**Definition 1.** *A tuple $(\boldsymbol{X}, \boldsymbol{y})$, where $\boldsymbol{X}$ is an $n \times p$ interval matrix and $\boldsymbol{y}$ is an $n \times 1$ interval vector, is called an (input for an)* ***interval regression model****. The* ***OLS-solution set*** *of $(\boldsymbol{X}, \boldsymbol{y})$ is*

$$B(\boldsymbol{X}, \boldsymbol{y}) := \{\beta : X^{\mathrm{T}}X\beta = X^{\mathrm{T}}y,\ X \in \boldsymbol{X},\ y \in \boldsymbol{y}\}.$$

The motivation for the definition is straightforward. Our aim is to use OLS to obtain an estimate of the unknown vector of regression parameters $\beta_0$ in the traditional linear regression model $\mathsf{E}(y) = X\beta_0$ (fulfilling traditional assumptions on error terms in order OLS be applicable). However, observations of both dependent variables ($y$) and independent variables ($X$) are interval-censored; i.e., we only know intervals $\boldsymbol{X}$ and $\boldsymbol{y}$ that are guaranteed to contain the directly unobservable data $(X, y)$. Then, the set $B(\boldsymbol{X}, \boldsymbol{y})$ contains *all possible* values of OLS-estimates of $\beta_0$ as $X$ and $y$ range over $\boldsymbol{X}$ and $\boldsymbol{y}$, respectively.

**Remark.** We *do not assume* any distribution on $\boldsymbol{X}$ or $\boldsymbol{y}$. Nevertheless, we may also regard $B(\boldsymbol{X}, \boldsymbol{y})$ in the following way: *if $X$ and $y$ are random variables such that the supports of the distributions of $X$ and $y$ are $\boldsymbol{X}$ and $\boldsymbol{y}$, respectively, then the support of the distribution of $(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$ is $B(\boldsymbol{X}, \boldsymbol{y})$.*

We take the liberty of neglecting the problem that $\boldsymbol{X}$ might contain a matrix that does not have full column rank. However, this problem is interesting from the computational point of view.

**Theorem 2.** *Deciding whether $B$ is bounded is a co-**NP**-complete problem.* □

**Proof.** Observe that if there a matrix $X \in \boldsymbol{X}$ does not have full column rank, then the set $\{\beta : X^{\mathrm{T}}X\beta = X^{\mathrm{T}}y\} \subseteq B$ is unbounded. On the other hand, let every $X \in \boldsymbol{X}$ be of full column rank. Then, by interval arithmetics [1], we get $\{\beta : X^{\mathrm{T}}X\beta = X^{\mathrm{T}}y\} \subseteq \boldsymbol{q}^{[X]}$, where

$$\boldsymbol{q}_i^{[X]} = [\sum_{j=1}^{n} \min\{Q_{ij}^{[X]}\underline{y}_j, Q_{ij}^{[X]}\overline{y}_j\}, \sum_{j=1}^{n} \max\{Q_{ij}^{[X]}\underline{y}_j, Q_{ij}^{[X]}\overline{y}_j\}], \quad i = 1, \ldots, p, \tag{1}$$

with $Q^{[X]} := (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$, which is a bounded set.

As any matrix $X \in \boldsymbol{X}$ is of full column rank, the set $\{\det X^{\mathrm{T}}X : X \in \boldsymbol{X}\}$ is compact and does not contain zero ($\det X^{\mathrm{T}}X$ being a continuous function on $\boldsymbol{X}$). By Cramer's Rule, each entry of $(X^{\mathrm{T}}X)^{-1}$ may be expressed as a fraction with $\det X^{\mathrm{T}}X$ in denominator. Thus, as $X$ ranges over $\boldsymbol{X}$, each entry of $(X^{\mathrm{T}}X)^{-1}$ ranges over a bounded set. Hence, also each entry of $Q^{[X]}$ ranges over a bounded set. Now it is easy to see that the set $\bigcup_{X\in\boldsymbol{X}} \boldsymbol{q}^{[X]} \supseteq B$ is bounded.

So we have shown that $B$ is unbounded if and only if $\boldsymbol{X}$ contains a matrix that does not have full column rank. By [10], that problem is $\boldsymbol{NP}$-complete. $\square$

From now on, we shall restrict ourselves to the case of $\underline{X} = \overline{X}$, the so-called *crisp input – interval output model* [7, 5]. We shall write simply $X$ instead of $\boldsymbol{X}$ if $\underline{X} = \overline{X}$.

## 2 In the crisp input – interval output model, the set $B$ is a zonotope

**Definition 3.** *The* ***Minkowski sum*** *of the set* $A \subseteq \mathbb{R}^k$ *and a vector* $g \in \mathbb{R}^k$ *is the set* $A \dotplus g := \{a + \lambda g : \ a \in A, \ \lambda \in [0, 1]\}$. *The* ***zonotope generated by*** $g_1, \ldots, g_N \in \mathbb{R}^k$ ***with shift*** $s \in \mathbb{R}^k$ *is the set*

$$\mathcal{Z}(s; g_1, \ldots, g_N) := (\cdots((\{s\} \dotplus g_1) \dotplus g_2) \dotplus \cdots \dotplus g_N).$$

*The* ***dimension*** *of the zonotope, denoted* $\dim(\cdot)$, *is the rank of* $\mathcal{L}(g_1, \ldots, g_N)$, *where* $\mathcal{L}(g_1, \ldots, g_N)$ *denotes the linear space generated by* $g_1, \ldots, g_N$.

The vectors $g_1, \ldots, g_N$ are called *generators*. Observe that the definition of dimension is correct in the sense that for any zonotope $Z$, $\dim(Z)$ is the largest $k'$ such that $Z$ contains a $k'$-dimensional ball (though $Z \subseteq \mathbb{R}^k$ with $k \geq k'$).

**Theorem 4.** *Let* $X \in \mathbb{R}^{n\times p}$ *be a matrix of full column rank and* $\boldsymbol{y}$ *an* $n \times 1$ *interval vector. Then*

$$B(X, \boldsymbol{y}) = \mathcal{Z}(Q\underline{y}; \ Q_1(\overline{y}_1 - \underline{y}_1), \ldots, Q_n(\overline{y}_n - \underline{y}_n)),$$

*where* $Q := (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$ *and* $Q_i$ *is the* $i$*-th column of* $Q$.

**Proof.**

$$
\begin{aligned}
B(X,\boldsymbol{y}) &= \{Qy : y \in \boldsymbol{y}\}\\
&= \{Q\underline{y} + Q\Delta : \Delta \in [0, \overline{y} - \underline{y}]\}\\
&= \{Q\underline{y} + Q\Delta : \Delta_i = \lambda_i(\overline{y}_i - \underline{y}_i), \lambda_i \in [0,1] \text{ for all } i = 1,\dots,n\}\\
&= (\cdots((\{Q\underline{y}\} \dotplus Q(\overline{y}_1 - \underline{y}_1, 0, \dots, 0)^{\mathrm{T}})\\
&\qquad \dotplus Q(0, \overline{y}_2 - \underline{y}_2, 0, \dots, 0)^{\mathrm{T}}) \dotplus \cdots) \dotplus Q(0,\dots,0,\overline{y}_n - \underline{y}_n)^{\mathrm{T}}\\
&= (\cdots((\{Q\underline{y}\} \dotplus Q_1(\overline{y}_1 - \underline{y}_1)) \dotplus Q_2(\overline{y}_2 - \underline{y}_2)) \dotplus \cdots) \dotplus Q_n(\overline{y}_n - \underline{y}_n)\\
&= \mathcal{Z}(Q\underline{y}; Q_1(\overline{y}_1 - \underline{y}_1), Q_2(\overline{y}_2 - \underline{y}_2), \dots, Q_n(\overline{y}_n - \underline{y}_n)). \quad \square
\end{aligned}
$$

There is a nice geometric characterization of zonotopes. Namely, a set $Z \subseteq \mathbb{R}^k$ is a zonotope if and only if *there exists a number $m$, a matrix $Q \in \mathbb{R}^{k\times m}$ and an interval $m$-dimensional vector $\boldsymbol{y}$ (i.e., $m$-dimensional cube) such that* $Z = \{Qy : y \in \boldsymbol{y}\}$. The interesting case is $m > k$. In that case we can say that zonotopes are images of "high-dimensional" cubes in "low-dimensional" spaces under linear mappings. In our setting, the set $B := B(X, \boldsymbol{y})$ is an image of $\boldsymbol{y}$ under the projection $Q = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$. We shall call $Q$ the *projection matrix.*

**Description of the set $B$.** In principle, there are (at least) four ways to describe the zonotope $B$:

(a) as the cube $\boldsymbol{y}$ and the projection matrix $Q$,

(b) as the set of generators and the shift vector,

(c) as an enumeration of vertices,

(d) as an enumeration of facets (i.e., a set of linear inequalities defining $B$ in $\mathbb{R}^p$).

The description (b) has been given in the Theorem; so we have

$$s = Q\underline{y}, \quad g_i = Q_i(\overline{y}_i - \underline{y}_i) \quad \text{for } i = 1,\dots,n.$$

The complexity of the descriptions (c) and (d) will be investigated in the next section. Some algorithms for enumeration of vertices are found in [2, 3].

# 3 Complexity of the vertex description and the facet description

**Theorem 5** ([13]). *For a zonotope $Z \subseteq \mathbb{R}^p$ with $n$ generators it holds $V(Z) \le 2\sum_{k=0}^{p-1}\binom{n-1}{k}$ and $F(Z) \le 2\binom{n}{p-1}$, where $V(Z)$ is the number of vertices and $F(Z)$ is the number of facets of $Z$. In general the bounds cannot be improved.* $\square$

We shall assume that $p$, the number of parameters of the interval regression model, is fixed. (In practice it is often the case that $p \ll n$.) Then we get the following order bounds for $V(Z)$ and $F(Z)$:

**Corollary 6.** *$V(Z) \leq O(n^{p-1})$ and $F(Z) \leq O(n^{p-1})$.* □

The Corollary implies that the number of vertices and the number of facets is polynomial in $n$. In [2] there is a general method for enumeration of vertices running in time polynomial in *size*(*input*) + *size*(*output*). The assumption that $p$ is fixed implies that *size*(*output*) is polynomially bounded by *size*(*input*). Hence:

**Corollary 7.** *If $p$ is fixed, then the vertex description of $B$ is computable in polynomial time.* □

Corollary 6 can be easily strengthened to the form $V(Z) \leq O(n^{\dim(Z)-1})$ and $F(Z) \leq O(n^{\dim(Z)-1})$. In statistical applications, this reduction usually does not help as it is rarely $\dim(B) < p$ (this could happen, for example, if a great majority of observations is crisp, i.e. if the cardinality of $\{i \in \{1, \ldots, n\} : \underline{y}_i < \overline{y}_i\}$ is smaller than $p$). However, there are important special cases where a significant reduction can be reached. The reduction is based on the following easy lemma.

**Lemma 8.** *Let $Z := \mathcal{Z}(s; g_1, g_2, \ldots, g_n)$ and for some $i$ and $j$, $i < j$ it holds $g_j = \alpha g_i$, where $\alpha \in \mathbb{R}$. Then*

$$Z = \begin{cases} \mathcal{Z}(s; g_1, \ldots, g_{i-1}, g_i + g_j, g_{i+1}, \ldots, g_{j-1}, g_{j+1}, \ldots, g_n) & \text{if } \alpha \geq 0, \\ \mathcal{Z}(s + g_j; g_1, \ldots, g_{i-1}, g_i - g_j, g_{i+1}, \ldots, g_{j-1}, g_{j+1}, \ldots, g_n) & \text{if } \alpha < 0. \end{cases}$$ □

The generator $g_j$ is called *redundant*. The process of removal of redundant generators may be iterated until all are removed; then we obtain a certain shift $s'$ and a reduced set of generators $g'_1, \ldots, g'_{n'}$ with $n' \leq n$ defining the same zonotope.

We can reformulate Theorem 5 in the following way.

**Corollary 9.** *Let $\sim$ be an equivalence on rows of $X$: $X_i \sim X_j$ iff $X_i$ is a multiple of $X_j$. Let $\nu$ be the number of equivalence classes of $\sim$. Then, $V(B) \leq 2\sum_{k=0}^{p-1}\binom{\nu-1}{k}$ and $F(B) \leq 2\binom{\nu}{p-1}$. In particular, if $p$ is fixed, then $V(B) \leq O(\nu^{p-1})$ and $F(B) \leq O(\nu^{p-1})$.* □

**Proof.** If $X_i$ is a multiple of $X_j$, then $Q_i(\overline{y}_i - \underline{y}_i)$ is a multiple of $Q_j(\overline{y}_j - \underline{y}_j)$ and we may apply Lemma 8. □

Observe that if the absolute term is involved in the regression model (i.e., $X$ contains an all-one column), then $X_i \sim X_j$ iff $X_i = X_j$. So, rather that saying that the combinatorial complexity of the zonotope $B$ depends on the number of observations, it is more appropriate to say that *the complexity depends on the number of distinct design points* (i.e. distinct rows of $X$). It is well-known that in practice regression problems with $\nu \ll n$ are quite frequent.

# 4 Usage of linear programming in analysis of algorithmic properties of the set $B$

The following Theorem shows some interesting complexity-theoretic facts about $B$. In the proof we use the well-known fact that linear programming is a polynomial-time solvable problem [8].

We first need a simple lemma.

**Lemma 10.** . *Each face of $B$ is center symmetric; in particular, $B$ itself is center symmetric. The center of $B$ is $Q\underline{y} + \frac{1}{2}\sum_{i=1}^{n} Q_i(\overline{y}_i - \underline{y}_i)$.*

**Proof.** The set $\{Q\underline{y}\}$ is center symmetric and the Minkowski sum preserves center symmetry. □

**Theorem 11.** *Let $X, \underline{y}, \overline{y}$ and $\beta$ be rational.*

*(a) We say that $\beta$ is **admissible** for $(X, \boldsymbol{y})$ if $\beta \in B(X, \boldsymbol{y})$. The question "is $\beta$ admissible?" is in $\boldsymbol{P}$, the class of problems decidable in Turing polynomial time.*

*(b) We say that $\beta$ is **extremal** for $(X, \boldsymbol{y})$ if $\beta$ is on the boundary of $B(X, \boldsymbol{y})$. The question "is $\beta$ extremal?" is in $\boldsymbol{P}$.*

*(c) The question "is $\beta$ a vertex of $B(X, \boldsymbol{y})$?" is in $\boldsymbol{P}$.*

**Proof.** For (a) observe that admissibility is decidable via the linear program

$$\max\ 0^{\mathrm{T}} y:\ \ \beta = Qy,\ \ \underline{y} \le y \le \overline{y}.$$

To prove (b) assume that the center of $B$ is 0 (by the previous Lemma, $B$ can be easily shifted). Then $\beta$ is extremal iff the optimal value of the linear program

$$\max\ w:\ \ w\beta = Qy,\ \ \underline{y} \le y \le \overline{y}$$

is $w = 1$.

In the proof of (c) we may assume that all generators $g_i := Q_i(\overline{y}_i - \underline{y}_i)$ are nonzero. Observe that $\beta$ is not a vertex iff there exists a generator $g_i$ such that $\beta$ can be shifted both in the direction $g_i$ and in the direction $-g_i$. So, $\beta$ is a vertex iff for each $i = 1, \ldots, n$ it holds that the linear program

$$\max\ w:\ \ \beta + wg_i = Qy,\ \ \beta - wg_i = Qz,\ \ \underline{y} \le y \le \overline{y},\ \ \underline{y} \le z \le \overline{y}$$

has the optimal value $w = 0$. □

Assume that $B$ is full-dimensional. Then, its volume is a natural measure of "fuzziness" of the regression model caused by interval censoring of the dependent variables. The statement (a) can be used for Monte Carlo estimation of the volume of $B$. It is easy to see that the interval vector $\boldsymbol{q}$ defined in (1) is the smallest interval vector containing $B$. So, we can generate a point $y \in \boldsymbol{q}$ at random and (a) provides a fast test for $y \in^{?} B$.

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