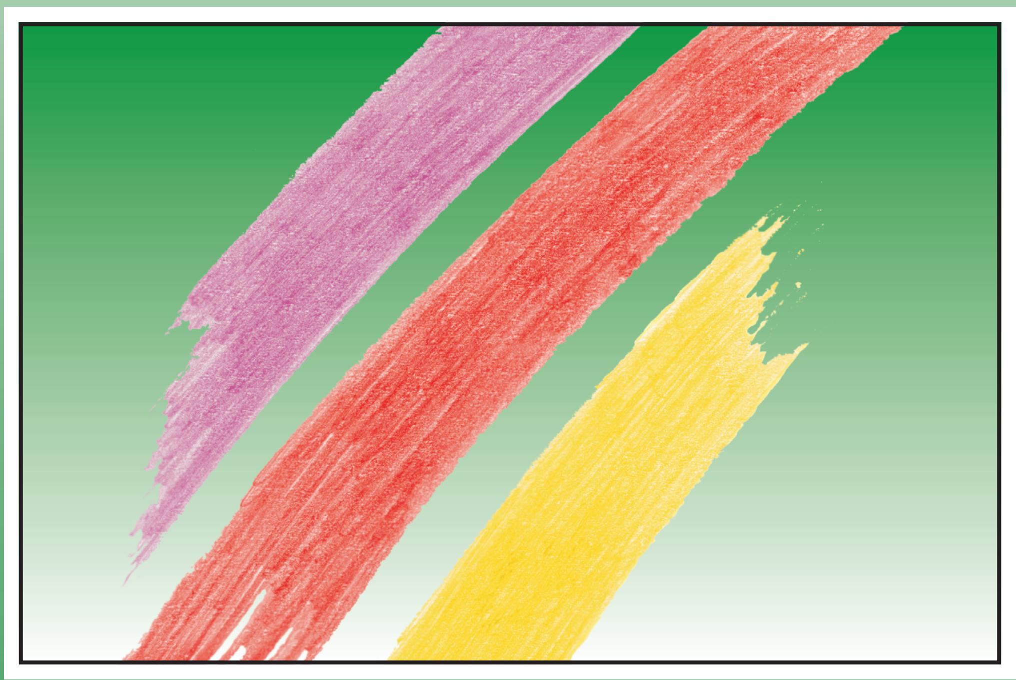
Quantitative Methods in Economics
(Multiple Criteria Decision Making XVI)



Proceedings of the International Scientific Conference
30th May - 1st June 2012
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CONTENT

MULTICRITERIA ASSESSMENT OF THE ERGONOMIC RISK PROBABILITY CREATION BY CHOSEN GROUPS OF STAKEHOLDERS WITH USING AHP METHOD WITHIN THE CONTEXT OF CSR

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ABSTRACT
Nowadays in the society more and more resounds the question about corporate social responsibility and sustainable competition of companies as well. The application of CSR (Corporate Social Responsibility) principles by achieving goals of company stakeholders imagines the significant milestone to reach upon mentioned sustainable competition. Despite of possible cross-understanding of stakeholders' interests on the company management side, shareholders and employees as well, constitutes the adaptation of employees work conditions a place for achieving of synergetic effect. For such an achieving of synergetic effect it is possible to use the ergonomics as a set of tools and methods which goal is to create the appropriate and ergonomics acceptable work environment. In terms of presented facts is this article handling with the multicriteria assessment of ergonomic risk probability creation using the AHP method (Analytic Hierarchy Process).

Keywords: *ergonomics, risk, multicriteral decision, sustainable development, corporate social responsibility*

JEL Classification: C02, C60
AMS Classification: 93A30, 97M40

Introduction

Changing of thinking paradigm of employers in relation to productivity can be achieved by creating appropriate working conditions for employees. Employees thus will be able to submit the required long-term work performance, through which the company can ensure a sustainable competitiveness on market. One of the most effective ways that company can declare a corporate social responsibility is just the application of ergonomics, which solution can reflect in a society-wide scale. Creating an ergonomically acceptable work environment recommend the monitoring of ergonomic risks, which resulting for example in the work environment, nature of work activities, physiological and psychological predispositions of employees. During watching the effects of ergonomic risks, it is necessary to measure and evaluate their acceptable level together with measuring the probability of their occurrence, and assess their effectiveness and mutual pairwise comparison of their effects.

1 ERGONOMICS AND ERGONOMICS RISK

Nowadays we have to deal with situation when companies need to fight with turbulent market conditions, influence of economic and monetary crisis, as well as moral crisis, as well as constantly changing and increasing of costumers demand and technological progress. The companies place more demands then ever before to increase employee productivity. With the increasing level of productivity is increasing the risk associated with the performance of work activities of employees and work environment influences and the level of the technique and technology.

The effective way to effectively prevent of risk occurrence is ergonomics, which represent a set of tools and methods, which can be used for its minimization.

Ergonomics risk (*R*) can be characterized as a relation between probability of the negative event and its consequences in general. It is a function of two basic parameters: probability (*P*) and consequence (*C*). We can express that fact with mathematical formula.

$$R = P.C \tag{1}$$

A number of factors of work conditions as well as whole work activities affect to workers, from which a many of them enter to the mutual interactions. We talk about their combined effect. The influence of the individual factors to human during the work can be for example, potentiate, add up, multiply, respectively decrease or eliminate. In general the ergonomics risk factors can be divided to [2]:

- ***modifiable risk factors*** – we can affect their direct effect to workers health and performance by preventive measures. More specifically it comes about physical, chemical, biological and psycho – social factors.
- ***unmodifiable risk factors or personality risk factors*** – we cannot affect their direct effect to workers health and performance: age, sex, body type, body dimensions.

In assessing of probability of ergonomic risks can be taken into account factors such as frequency of risk factor occurrence, size of resulting consequences, time exposition, possibilities of using of protective measures, opportunity of avoiding given risk, or factors like speed of risk event creation, awareness of the risk itself and possibility of prevention.

Quantitative assessment of ergonomics risk requires defining level of acceptability of risk, which is many times the subject of extensive discus. The relation between the probability of risk factor impact to creation of threats to health and workers work well-being and possible outcome of it influence, i.e. the severity of possible damage to workers health is in the following table [1].

Table 1 Assess of risk degree

	P1	**P2**	**P3**	**P4**	**P5**
D1	**H**	**H**	**H**	**H**	**H**
D2	**H**	**H**	**H**	**H**	**H**
D3	**H**	**H**	**M**	**M**	**M**
D4	**H**	**M**	**M**	**L**	**L**
D5	**M**	**M**	**L**	**L**	**L**

Probability of occurrence
P1 Almost certain
P2 very probable
P3 Not usual, but possible
P4 Low probable
P5 Practically improbable

Possible consequence of
D1 Fatal, permanent inability to work
D2 Need for long term treatment
D3 Short term treatment
D4 Need of longer rest
D5 Need of shorter rest

2 STAKEHOLDERS AND CSR

Today, the time of turbulent market and social changes requires every business have to realize its responsibility towards society. The concept of corporate social responsibility (CSR) is up-to-date business philosophy implementing which company tries to reach sustainable development and related to sustainable competitiveness, sustainable production sustainable consumption, etc.

Although there's no one generally accepted definition of CSR, the definition from European Commission (Green Paper, 2001) claimed "*CSR is a concept whereby companies integrate social and environmental concerns in their business operations and in their interaction with their stakeholders on a voluntary basis"* [3], it refers to stakeholders as the necessary part of the CSR issues.

Stakeholder concept mentioned in the definition is a basic topic of CSR because of stakeholders are all those that are in relation to company. Stakeholders can be individuals, group of individuals or organizations that have any effect on business activities or they are affected by the company on the contrary.

Stakeholders are considered as a group of people with a recognizable relationship to the company including: shareholders, employees, customers, suppliers and distributors, local communities, competition, media, public, NGOs, government, other business partners.

Employees are one of the most important internal stakeholders because of time, energy and efforts they put to company to reach the success and sustainable competitiveness. The relationship between employee and company is considered to be important by society, because employees contribute their efforts and time towards the development of company, which in turn improver society. In return of their work employees' not only expect wages, but also security and proper working conditions to do their job in friendship and healthy environment. Some specific responsibilities of organization towards their employees are [4]:

- ***to provide adequate compensation***,
- ***to ensure open and honest communication*** with employees that respect each employee's health and dignity,
- ***to encourage and assist employees*** in developing skills and knowledge that are required for accomplishing the task.
- ***to listen and act to employees' requests, suggestion, ideas and complaints*** wherever possible,
- ***to generate equal treatment and opportunity*** regardless of gender, age, race and religion,
- ***to provide optimal working conditions that must be ergonomically acceptable***.

Both business and employees have certain commitments towards each other. To support company in reaching sustainable competitiveness, company should maintain an ergonomically healthy work environment, where the employees fulfil their responsibilities.

3 AHP METHOD

AHP method is one of the multicriteria optimization methods and exact methods as well. This method can be used within most varied situations where an optimal alternative is searched and a lot of factors are influencing on these possible alternatives – criteria.

According to the author of this method Saaty, is composed of three parts:

1. **Hierarchy** – goals, groups of experts, criteria and alternatives are sorted in a hierarchical structure like a tree diagram form and apply a principle that it is going forward from the general to the specifications.
2. **Priorities** – the AHP method is also based on a pairwise comparison always of the two elements between each other, i.e. through the priorities are compared criteria and we are looking for the most important criterion and for the optimal alternative as well. This is done using the scale like it is described in the table 2.

The priorities are recorded into the Saaty's matrix where elements located on a diagonal have a value 1. Reciprocity rule applies: $s_{ij} = 1/s_{ji}$. (2)

$$S = \begin{pmatrix} 1 & s_{12} & s_{13} & \dots & s_{1n} \\ 1/s_{12} & 1 & s_{23} & \dots & s_{2n} \\ 1/s_{13} & 1/s_{23} & 1 & \dots & s_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1/s_{1n} & 1/s_{2n} & 1/s_{3n} & \dots & 1 \end{pmatrix}$$

Table 2 AHP fundamental scale

Intensity of importance	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective.
3	Moderate Importance	Experience and judgement slightly favour one activity over another.
5	Strong Importance	Experience and judgement strongly favour one activity over another.
7	Very strong or demonstrated importance	An activity is favoured very strongly over another; its dominance demonstrated in practice.

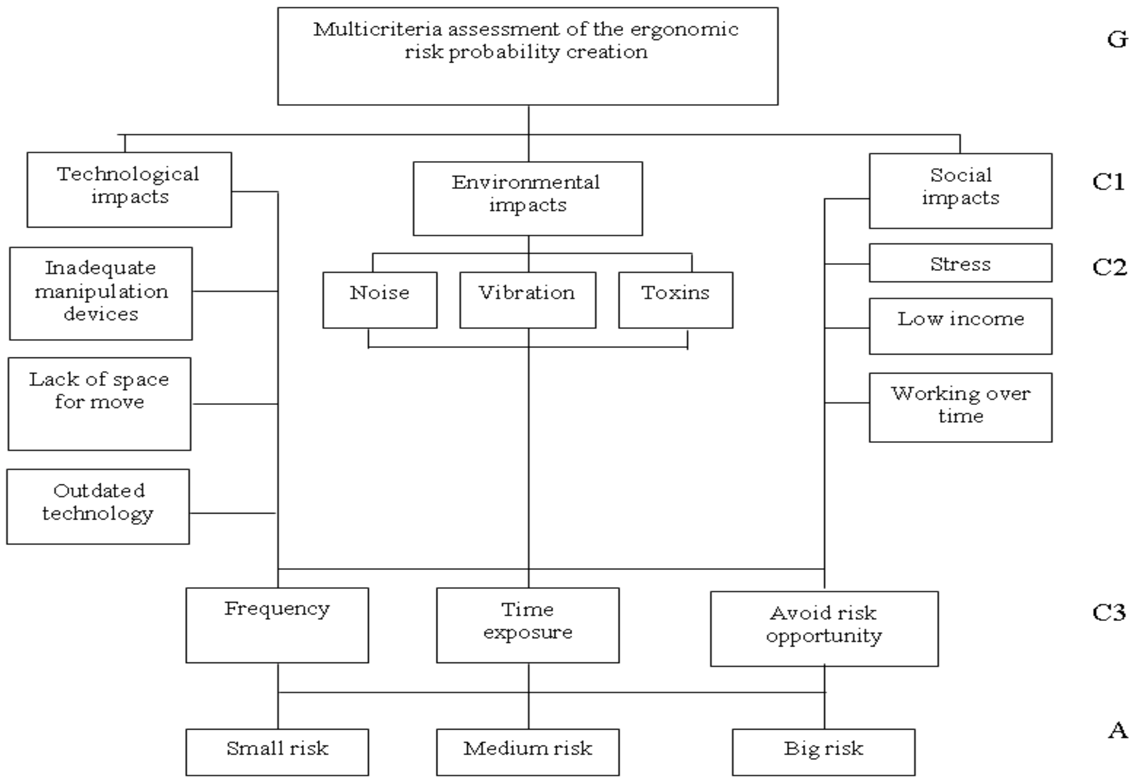
9	Extreme Importance	The evidence favouring one activity over another is of the highest possible order of affirmation.
2,4,6,8	Intermediate values between 2 adjacent scale values	Compromise is needed between two levels.

3. **Consistency** – is something like a test of accuracy where we are examining if a pairwise comparison and the final matrix and a result as well are consistent. This is done according to the CI indicator – consistency index $CI = (\lambda_{\max} - n)/(n-1)$. (3) Where "n" is a matrix size.

4 **Multicriteria assessment of the ergonomic risk probability creation by chosen groups of stakeholders with using AHP method within the context of CSR**

For the purpose of this article we have created an illustrative example of multicriteria assessment of the ergonomic risk probability creation by chosen groups of stakeholders using the AHP method within the CSR context – goal ***G***.
As a method for multicriteria decision making was chosen above mentioned AHP method. Ergonomic risk like it is in the table 1 described can take values named by abbreviations: ***S – small, M – medium, B – big***.
Selected group of stakeholders are in this case employees of the manufacturing company which most influence ergonomic risks have.
Connection with CSR concept is done by the main level of criteria – ***C1***. Technological impact – are hard indicators of the CSR economical pillar, environmental impacts – are indicators of the environmental pillar and last but not least social impacts which are indicators of the social pillar acting on employees. The main criteria level is further divided into the subcriteria – ***C2*** specific for each impact and these are also further divided into the last three criteria ***C3***: frequency, time exposure and avoid risk opportunity.
Alternatives ***A*** of the assessment are risk values as described above small, medium and big. Hierarchical structure of this illustrative model is illustrated on the figure 1.

Figure 1 Hierarchical structure of the illustrative model



A software tool for AHP method solution is a program called Expert Choice which is possible to use also in this illustrative model. Input priority to each criterion should be done by a group of experts using the questionnaire. For the purpose of our article we describe illustrative the final solution using the EC with randomly given priorities. This means that for every case used with the help of this model will be in each company different point of view on values and priorities given for criteria and alternatives. On the figure 2 is illustrated a criteria pairwise comparison between each other and the next figure 3 illustrates the optimal alternative according to the all mentioned criteria.

Figure 2 Criteria comparison

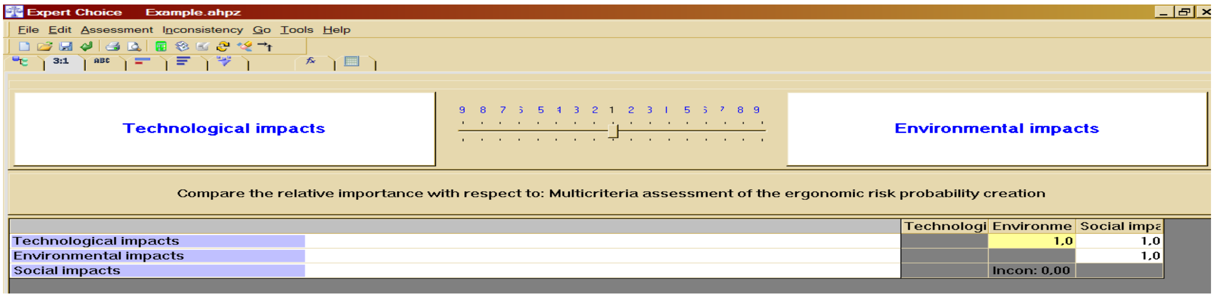
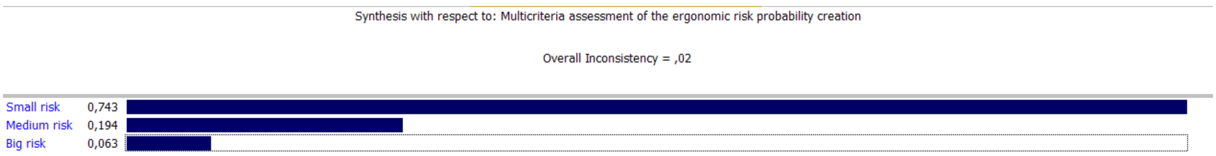


Figure 3 View of the optimal alternative



In our model example was using the EC software detected that by the model values would be the small risk the optimal alternative. Nevertheless as already mentioned above, priority values are depend on each company.

Conclusions

The presented article focused on the possibility of using exact methods, respectively of using multi-criteria decision making for practical use in business practice. Through brief modeled situation we pointed out their use in the ergonomics area in terms of selected groups of stakeholders within the context of corporate social responsibility

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THE MULTIPLE CRITERIA DECISION MAKING METHOD COMPUTATIONALLY BASED ON THE ASSIGNMENT PROBLEM

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Abstract
The paper introduces the multiple criteria decision making method using one of basic problems of linear programming as its computational principle. Thus the method algorithm is based on the assignment problem enabling matching alternatives and rank. The weights of all evaluative criteria set by decision maker are demanded as input information. The approach is proposed in two modifications. One takes into account the differences among criterial values the other no. Finally apart from the theoretical principles the article offers some practical application in investment decision making process in the field of capital market with shares funds.

Keywords: *assignment problem, difference, MCDM method, shares fund*

JEL Classification: C63, G11
AMS Classification: 90-08

1 INTRODUCTION

In this article a reader can see two important missions. Firstly we propose the multiple criteria decision making method based on the assignment problem. The basic idea is described in some formally modified form compared to (Bouška et al., 1981). This approach does not take into account the differences among criterial values, thus the method modification will be projected. Secondly we can meet the application of proposed method in terms of capital market. Some investor wants to choose suitable investment shares fund[1] from the offer of Investment company Česká spořitelna. We will obtain two different results and compare them.

2 THE MCDM METHOD MAKING USE OF THE ASSIGNMENT PROBLEM

The assignment method is described in (Bouška et al., 1984 or Hwang et al., 1981). We will use a little (rather) formally modified basic algorithm and then propose deeper modification in order to afflict the importance of criterial values. Firstly we look at the basic idea of this method in the following several steps:

First step: Observe the matrix of valuations of all alternatives according to particular criteria $Y = (y_{ij})$, where y_{ij} $(i = 1, 2, ..., p;\ j = 1, 2, ..., k)$ represents evaluation of i^{th} variant by j^{th} characteristic. Given the vector of criteria weights $v_j = (v_1, v_2, ..., v_k)$ defined by decision maker in agreement with his preferences[2].

Second step: For each alternative we create the ranking vector $a_i = (a_{i1}, a_{i2}, ..., a_{ik})$, whereas the component a_{ij} $(i = 1, 2, ..., p; j = 1, 2, ..., k)$ presents rank of i^{th} variant according to j^{th} criterion. In virtue of this vector we identify the set A_{ij} $(i, j = 1, 2, ..., p)$ containing the indexes of criteria by which j^{th} alternative is placed in i^{th} position in the case of no indifferent references according to all tracked characteristics. Under the indifferent relations specify the sets E^n_{ij} $(i, j = 1, 2, ..., p)$ including the indexes of criteria by which j^{th} alternative shares i^{th} position with the n others[3].

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[1] *Shares fund* is inside organizational entity of investment company without legal identity (Valach, 2006).
[2] The weight vector can be established on the basis of the points method (see more Fiala, 2008).
[3] For instance: When j^{th} alternative shares 5^{th} place with two others by k^{th} criterion, so we create three sets for j^{th} variant as $E^3_{5j} = E^3_{6j} = E^3_{7j} = \{k\}$.

Third step: Design the matrix $B = (b_{ij})$, where it must hold

$$b_{ij} = \sum_{g \in A_{ij}} v_g + \sum_{g \in E_{ij}^n} \tfrac{v_g}{n} \quad i, j = 1, 2, ..., p. \tag{1}$$

Thus the item b_{ij} characterizes the sum of weights corresponding to such a criteria that assign i^{th} place to j^{th} alternative. In the case of indifferent relations among alternatives by concrete criteria, the component b_{ij} does not include a whole value of weight v_g, but only its relative part v_g / n, where n denotes the number of variants placed in the same position. The value b_{ij} actually expresses the fitness of assignment of i^{th} position to j^{th} alternative.

Fourth step: The final ranking of alternatives is set on the basis of following assignment problem[4]

$$z = \sum_{i=1}^{p} \sum_{j=1}^{p} b_{ij} x_{ij} \rightarrow \max$$

$$\sum_{i=1}^{p} x_{ij} = 1 \quad j = 1, 2, ..., p$$

$$\sum_{j=1}^{p} x_{ij} = 1 \quad i = 1, 2, ..., p \tag{2}$$

$$x_{ij} = \{0, 1\} \quad i, j = 1, 2, ..., p,$$

where x_{ij} takes the value 1 if the j^{th} variant is placed in i^{th} place, otherwise equals 0. As we endeavour after the most efficient assignment in the sense values b_{ij}, the objective function is maximized.

The basic form of the assignment method provides full ranking of alternatives. The indisputable advantage of this method is that the criterial values must not be standardized. On the other side we should not forget that the differences among alternative's criterial values by particular criterion are not taken into account in described algorithm.

3 THE MODIFICATION OF THE ASSIGNMENT METHOD

As mentioned above, the modified algorithm of method accepts the distances among criterial values. So this approach becomes more complicate, eventually in virtue of data standardization. The modified method principle will be introduced during a few following steps.

First step: Observe again the matrix of criterial values $Y = (y_{ij})$ with size p x k and input information as weight vector $v_j = (v_1, v_2, ..., v_k)$. At the first transform all minimized criteria to maximized form according to

$$q_{ij} = \max_i (y_{ij}) - y_{ij} \quad \forall j \text{ (min)},$$

$$q_{ij} = y_{ij} \quad \forall j \text{ (max)}. \tag{3}$$

As in the basic algorithm create the ranking vector $a_i = (a_{i1}, a_{i2}, ..., a_{ik})$, whereas the component a_{ij} $(i = 1, 2, ..., p; j = 1, 2, ..., k)$ presents rank of i^{th} variant according to j^{th} criterion and then the set A_{ij} $(i, j = 1, 2, ..., p)$ containing the indexes of criteria by which j^{th} alternative is placed in i^{th} position in the case of no indifferent references according to all watched characteristics. Under the indifferent relations we specify the sets E_{ij}^n $(i, j = 1, 2, ..., p)$ including the indexes of criteria by which j^{th} alternative shares i^{th} position with the n others.

[4] The basic idea of assignment problem can be described as mutual assignment of elements from two stated sets that should be as efficient as possible, it means with minimum costs, maximum returns etc. (Jablonský, 2007)

<u>Second step:</u> Create the sets for $i = 1,2,...,p,\ j = 1,2,...,k$

$$C_{ij}^{l} = \{r, q_{ij} > q_{rj}; r = 1,2,...,p, i \neq r\}, \tag{4}$$

$$\text{or } C_{ij}^{h} = \{r, q_{ij} < q_{rj}; r = 1,2,...,p, i \neq r\} \tag{5}$$

containing all indexes of alternatives *r* which are evaluated worse, or better than i^{th} variant by j^{th} criterion. Then we can define two matrixes $Q^{l} = (q_{ij}^{l})$ and $Q^{h} = (q_{ij}^{h})$, where

$$q_{ij}^{l} = \frac{\sum_{r \in C_{ij}^{l}} (q_{ij} - q_{rj})}{\left|C_{ij}^{l}\right|} \quad i = 1,2,...,p \quad j = 1,2,...,k,$$
$$q_{ij}^{h} = \frac{\sum_{r \in C_{ij}^{h}} (q_{rj} - q_{ij})}{\left|C_{ij}^{h}\right|} \quad i = 1,2,...,p \quad j = 1,2,...,k \tag{6}$$

expressing the average distance from alternatives which are evaluated worse, or better than i^{th} alternative by j^{th} criterion. The expressions $\left|C_{ij}^{l}\right|, \left|C_{ij}^{h}\right|$ denote the cardinality of sets C_{ij}^{l}, or C_{ij}^{h}.

<u>Third step:</u> Now the values q_{ij}^{l} and q_{ij}^{h} must be standardized as follows

$$t_{ij}^{l} = \frac{q_{ij}^{l}}{\max_{i}(q_{ij}^{l})} \quad i = 1,2,...,p \quad j = 1,2,...,k,$$
$$t_{ij}^{h} = \frac{q_{ij}^{h}}{\max_{i}(q_{ij}^{h})} \quad i = 1,2,...,p \quad j = 1,2,...,k. \tag{7}$$

<u>Fourth step:</u> Design the matrixes $B^{h} = (b_{ij}^{h})$ and $B^{d} = (b_{ij}^{d})$, where the following formulas must hold

$$b_{ij}^{l} = \sum_{g \in A_{ij}} v_g t_{jg}^{l} + \sum_{g \in E_{ij}^{n}} \tfrac{v_g}{n} t_{jg}^{l} \quad i, j = 1,2,...,p,$$
$$b_{ij}^{h} = \sum_{g \in A_{ij}} v_g t_{jg}^{h} + \sum_{g \in E_{ij}^{n}} \tfrac{v_g}{n} t_{jg}^{h} \quad i, j = 1,2,...,p. \tag{8}$$

The components b_{ij}^{l}, or b_{ij}^{h} $(i, j = 1,2,...,p)$ represents the sum of weighted average distances from worse, or better alternatives in the case of assignment of j^{th} evaluated alternative to i^{th} place by all criteria. In the case of indifferent relations among alternatives by concrete criteria, the component b_{ij} does not include a whole value of weight v_g, but only its relative part v_g / n, where *n* denotes the number of variants placed in the same position as in the basic algorithm. We can say again the greater b_{ij}^{l}, or lower b_{ij}^{h} the major fitness of the assignment of j^{th} alternative to i^{th} position.

If there are no assignment of i^{th} order to j^{th} variant according to all criteria, the values b_{ij}^{l} will be low enough (e. g. -10), on the contrary b_{ij}^{h} must be positive (e. g. 10) or at least equal zero as follows

$$b_{ij}^{l} = -10 \quad i, j = 1,2,...,p \quad A_{ij}, E_{ij}^{n} = \varnothing,$$
$$b_{ij}^{h} = 10 \quad i, j = 1,2,...,p \quad A_{ij}, E_{ij}^{n} = \varnothing. \tag{9}$$

These values are stated in sufficient size in comparison with other elements in the matrixes B^{l} and B^{h} not to take place unsolicited assignment of i^{th} place to j^{th} alternative.

<u>Fifth step:</u> We apply the linear problem again for creation the final alternative ranking. But this time we use two assignment problems which are able to do order by average deviations from worse and also better alternatives. Formulate as

$$z_1 = \sum_{i=1}^{p}\sum_{j=1}^{p} b_{ij}^l x_{ij}^l \rightarrow \max \qquad z_2 = \sum_{i=1}^{p}\sum_{j=1}^{p} b_{ij}^h x_{ij}^h \rightarrow \min$$
$$\sum_{i=1}^{p} x_{ij}^l = 1 \quad j = 1,2,...,p \qquad \sum_{i=1}^{p} x_{ij}^h = 1 \quad j = 1,2,...,p$$
$$\sum_{j=1}^{p} x_{ij}^l = 1 \quad i = 1,2,...,p \qquad \sum_{j=1}^{p} x_{ij}^h = 1 \quad i = 1,2,...,p \tag{10}$$
$$x_{ij}^l = \{0,1\} \quad i,j = 1,2,...,p \qquad x_{ij}^h = \{0,1\} \quad i,j = 1,2,...,p,$$

where x_{ij}^l, or x_{ij}^h takes 1 if the alternative j is placed in i^{th} position, otherwise equals 0. As we try to reach the assignment as efficient as possible in the sense of values b_{ij}^l, or b_{ij}^h, the objective function z_1 is maximized, z_2 minimized.

Sixth step: Given two alternative ranks have not to be the same. The whole order will be obtained by the help of simple mean of both ranking as follows

$$k_j = \frac{d_j^l + d_j^h}{2}, \tag{11}$$

whereas d_j^l and d_j^h is the fractional place of j^{th} variant according to assignment problems mentioned above. The final rank is stated on the basis of descending ordered values k_j.

The described method takes into account the differences among the criterial values unlike the basic form. On the other side its computational procedure becomes more complicated. The final order can also contain indifferent relations.

4 PRACTICAL APPLICATION

The investor wants to insert his free financial resources in some open shares funds. As a long term client of Česká spořitelna he chooses the shares funds provided by Investment company Česká spořitelna offering four basic groups of funds, namely *money-market funds*, *mixed funds*, *bond funds* and *stock funds*. The list of these funds is showed in the following table (Tab. 1).

Table 1 The list of shares funds offered by Investment company Česká spořitelna. Source: Self-designed in MS Excel[5]

Money-market funds	Mixed funds	Bond funds	Stock funds
Sporoinvest	*Osobní portfolio 4 Plus* *Fond řízených výnosů* *Konzervativní MIX* *Vyvážený MIX* *Dynamický MIX* *Akciový MIX*	*Sporobond* *Trendbond* *Bondinvest* *Korporátní dluhopisový* *High Yield dluhopisový*	*Sporotrend* *Global Stocks* *Top Stocks*

The investor wants to gain the ranking of all tracked open shares funds in order to make the right investment decision. Three evaluative criteria are set by the investor (decision maker), in the concrete *return*[6], *riskiness*[7] and *costs*[8]. He determines the weights of all criteria by the help of the points method as follows (Tab. 2).

[5] http://www.iscs.cz/web/fondy/ (cit. 30. 1. 2012)
[6] We take into account average monthly returns from 1st April 2009 to 1st December 2011. This period had to be cut for mixed shares funds *Osobní portfolio 4* and *Plus* because of their later foundation.
[7] The risk is stated as a standard deviation of fund returns.
[8] The costs include the entry fees.

Table 2 The weights of criteria. Source: The supplement Sanna in terms of MS Excel

Return	Riskiness	Costs
0,38	0,48	0,14

If we apply the basic form of described MCDM method we will obtain the following rank.

Table 3 The final ranking of shares funds by the basic form of MCDM method. Source: The optimization software LINGO

1.	*Sporoinvest*	**9.**	*Vyvážený Mix*
2.	*Fond řízených výnosů*	**10.**	*Korporátní dluhopisový*
3.	*Osobní portfolio 4*	**11.**	*Dynamický Mix*
4.	*Plus*	**12.**	*Global Stocks*
5.	*Sporobond*	**13.**	*Akciový Mix*
6.	*Konzervativní Mix*	**14.**	*High Yield dluhopisový*
7.	*Bondinvest*	**15.**	*Top Stocks*
8.	*Trendbond*	**16.**	*Sporotrend*

As we can see above, the first investment alternative is represented by *Sporoinvest*. This one embodies the best values of criteria riskiness and costs. Despite so bad return, it is the number one. On the other end there are stock shares funds by virtue of their very large rate of riskiness and costs. Their very good return did not help to their better rank.
Now we try to use the modified algorithm, then the result is in Tab. 4.

Table 4 The final ranking of shares funds by the modified form of MCDM method. Source: The optimization software LINGO

1.	*Sporoinvest*	**9.**	*Korporátní dluhopisový*
2.	*Fond řízených výnosů*	**10.**	*Vyvážený Mix*
3.	*Sporobond*	**11.**	*High Yield dluhopisový*
4.	*Trendbond*	**12.**	*Dynamický Mix*
5.	*Konzervativní Mix*	**13.**	*Global Stocks*
6.	*Osobní portfolio 4*	**14.**	*Top Stocks*
7. - 8.	*Bondinvest*	**15.**	*Akciový Mix*
7. - 8.	*Plus*	**16.**	*Sporotrend*

It is obvious that the ranking is not the same as previous. However the first place was not threatened, the last one as well. The biggest shift was noted by the shares funds *Trendbond* and *Plus*. It is caused thanks differences among criterial values. The fall of fund *Plus* is influenced by so long distances from the better alternatives due to solitary negative return, on the contrary the rise of fund *Trendbond* reflects low differences from better alternatives, especially in risk and costs, despite of comparatively worse order.

5 CONCLUSION

This paper offers two concept of multiple criteria decision making method based on the assignment problem in computational procedure. The modification of the basic algorithm was proposed in order to implicate the differences among criterial values in the computational algorithm. From the practical point of view the results, in accordance with both methods described above, are expectantly different. Finally we cannot omit that the second method may offer final ranking with indifferent references which would be taken as some disadvantage. This fact can be (e. g.) removed by applying a principle with only one assignment problem (and with connected consequences), thus either take into account the deviations from worse, or better alternatives.

Acknowledgements
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STOCHASTIC INVENTORY LOCATION MODEL

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Abstract
Paper presents stochastic version of the location model incorporating inventory cost in facility location decision making. The goal is to optimize location, inventory, and allocation decisions under stochastic demand described by discrete scenarios. Objective function of proposed model minimizes the expected total cost, including location, transportation and inventory costs of distribution system over all scenarios.

Keywords: *stochastic demand, facility location, inventory, optimization*

JEL Classification: C44
AMS Classification: 90B80

INTRODUCTION

Logistics technologies and logistics operations are based on the interaction of various subsystems of the distribution system. Its optimization offers the opportunity to achieve desired results. A logistics technology is possible to describe as a sequence of decision-making processes and procedures, respecting the logistical interaction between the components of the logistics system in given economic environment. Under the term optimization of processes we can understand optimization of raw materials, its storage and release, the smooth flow of production and its semi-finished products. Goal is to ensure the movement of material from point of origin to point of consumption to meet the needs of end customers.
Decision making about distribution system design or redesign is based on given database of information. The main goal of distribution system design or redesign is to minimize costs. These costs include costs of purchasing, production costs, inventory costs, costs of individual facilities, transportation costs and costs associated with different level of service requirements. Suppose that the cost of opening distribution center are fixed costs (for example construction costs, rental costs, etc.) and to variable costs belongs transport costs, storage costs, etc.. Requirements on the level of customers service leads to increase of inventory costs, due to increase of safety stocks, increase in opening and operation costs and transportation costs of goods distributing from warehouses to the end customer. Therefore, decisions about distribution system design are important factors significantly affecting the efficiency of the distribution system.
In decision making process we predict some elements of system, but we cannot predict them with certainty. The physical form of the distribution system is not static and must be adapted to changes such as changing customer demand, changes in product mix, supply strategies or fluctuations in the cost of equipment, thus changes in the tactical level. In this case even deterministic models contain elements of randomness, therefore their future values can be predicted only with some probability.

2 STOCHASTIC INVENTORY LOCATION MODEL

The aim of our model is to minimize the total costs of setting up, transporting, holding and ordering inventory. Therefore, our model is divided into four main parts. The first part is the sum of the fixed installation costs. Company pays for the construction of DC. The second part is the transportation costs associated with transporting supplies from the manufacturer to DC for a time period of one year. Inventory costs are shown in the third section. The last part represents the cost of safety stock.

2.1 Notation

We use the following notation:

Sets

I set of retailers, indexed by i

J set of potential , indexed by j

S set of scenarios, indexed by s

Parameters

Costs

Co_j fixed cost per year of opening DC j, for $j\epsilon J$

Ct_{ij} per unit annual cost for transport

Cs per unit cost for order

Ca fixed cost for order

Demand

λ_{is} mean annual demand at retailer i in scenario s, for $i\epsilon I, s\epsilon S$

σ_{is}^2 variance of lead time

θ_{is}^2 variance of demand at retailer i in scenario s, for $i\epsilon I, s\epsilon S$

L_j lead time from DC to retailer

z safety coefficient, customer service level

Decision Variables

$$X_j = \begin{cases} 1, & \text{if is opened DC } j \in J \\ 0, & \text{otherwise} \end{cases}$$

$$Y_{ijs} = \begin{cases} 1, & \text{if retailer } i \in I \text{ is served by DC } j \in J \text{ in scenario } s \in S \\ & 0, \quad otherwise \end{cases}$$

Location decisions X are scenario-independent: they must be made before it is known which scenario will be realized. Assignment decisions are scenario-dependent, so the Y variables are indexed by scenario. Inventory decisions are also scenario-dependent, in that the levels of safety stock change once assignments are made and demand means and variances are known, though there are no explicit inventory variables.

3.2 Formulation

In this paper we formulate stochastic facility location model with know probability distribution for each scenarioρ_s, with condition $\sum_{s=1}^{S} \rho_s = 1$. Than objective function minimizing above mentioned costs looks like:

$$\min \sum_{s=1}^{S} \sum_{j=1}^{J} \rho_s \left(Co_j X_j \sum_{i=1}^{I} \lambda_{is} Ct_{ij} Y_{ijs} + Cs \sqrt{\frac{2Ca \sum_{i=1}^{I} \lambda_{is} Y_{ijs}}{Cs}} + Ca \frac{\sum_{i=1}^{I} \lambda_{is} Y_{ijs}}{\sqrt{\frac{2Ca \sum_{i=1}^{I} \lambda_{is} Y_{ijs}}{Cs}}} + zCs \sqrt{\sum_{i=1}^{I} L_j \sigma_{is}^2 Y_{ijs} + \sum_{i=1}^{I} \lambda_{is} \theta_{is}^2 Y_{ijs}} \right) \quad (1)$$

subject to

$$\sum_{j=1}^{J} Y_{ijs=1} \qquad i = 1,2, \dots I, s = 1,2, \dots S \quad (2)$$

$$Y_{ijs} \le X_j \qquad i = 1,2, \dots I, j = 1,2, \dots J, s = 1,2, \dots S \quad (3)$$

$$X_j \in \{0,1\} \qquad j = 1,2,\ldots J \tag{4}$$

$$Y_{ijs} \in \{0,1\} \qquad i = 1,2,\ldots I, j = 1,2,\ldots J, s = 1,2,\ldots S \tag{5}$$

The objective function (1) value represents the expected value of the individual-scenario costs. The first term inside the parentheses computes the fixed cost of locating DCs. The second term computes the expected cost to transport goods from the supplier to the DCs as well as the variable shipment cost from the DCs to the retailers. The third term computes the expected cost of holding working inventory at the DCs, assuming that each DC follows an economic order quantity (EOQ) policy, including the fixed costs of shipping from the supplier to the DCs. This term is similar to the classical expression for the optimal EOQ cost given by $\sqrt{\frac{2\lambda Ca}{Cs}}$ and fixed costs of order $Ca\frac{\lambda}{Q^*}$. Finally, the fourth term represents the expected cost of holding safety stock at the DCs. This term represents cost of safety inventory hedging against randomness in future customer demands and lead time. Constraints (2) require each retailer to be assigned to exactly one DC in each scenario. Constraints (3) prohibit a retailer from being assigned to a given DC in any scenario unless that DC has been opened. Constraints (4) and (5) are standard integrality constraints.

3 CONCLUSION

Facility location model with inventory cost integration is proposed in this paper. Its aim is to minimize costs of operation of distribution system. Stochastic aspect of model is lead time in itself and in demand of customers during the lead time.

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MODELING OF RETURN IN REVERSE LOGISTICS

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Abstract
Main topic of reverse logistics is return of end-of-life product from the point of final consumption back to the manufacturer. This return can be facilitated by wide range of subjects and in different ways. There are many different approaches how this return can be calculated because there is a number of factors, including the life-cycle stage of a product and the rate of technological change, which influence the quality and quantity of the returns.
These characteristics have major impact on demand management, and inventory control. The high level of uncertainty arising from different characteristics of quantity and quality of the returned products. This makes the production planning task more complicated and demands different methods of calculating the return.

Keywords: *Recycling, Return, End-of-Life Products, Calculation*

JEL Classification: C44
AMS Classification: 90C15

INTRODUCTION

Along with the notion of reverse logistics the questions of extending products' life cycle for next reuse, remanufacturing or recycling of used materials has Has become more important (Kostecki, 1998). New approaches in the field of reverse logistics have been developed (Stock, 1998) and many examples of how to use knowledge of reverse logistics practices have arisen (Rogers and Tibben-Lembke, 1999; Guide, 2000).
This paper describes the possibility of obtaining the materials through recycling which can generate additional cost savings associated with the purchase of new materials needed for production. It explains the importance of recycling, not only for environmental and social reasons but also for economic reasons as well. We also describe the ways how the used products return back to the manufacturer in time.
The paper is involved in modeling of the scenarios for the return of products depending on time period from which the products are returned back to the manufacturer.
Processes associated with the products return and its reuse or remanufacturing are influenced by many factors related to its lifecycle (e.g. service, maintenance, redesign, upgrade, redesign or change the packaging, remanufacturing, reusing, recycling and disposal). The management of these factors is associated with a new kind of uncertainty caused by the nature of the return process - uncertainty about the number and quality of the returned product.
There are seven characteristics of the recoverable manufacturing systems that complicate the management, planning, and control of supply chain functions. They are:[1]

- The uncertain timing and quantity of returns,
- The need to balance demands with returns,

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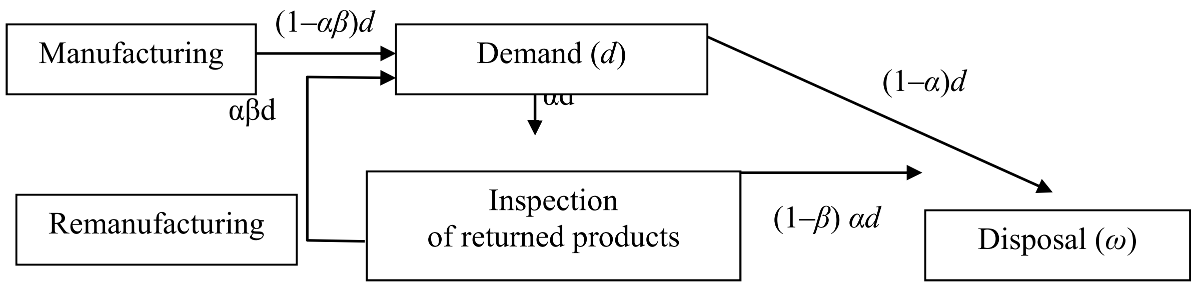
- The need to disassemble the returned products,
- The uncertainty in materials recovered from returned items,
- The requirement for a reverse logistics network,
- The complication of material matching restrictions,
- The problems of stochastic routings for materials for repair and remanufacturing operations and highly variable processing times.

1 RETURN OF PRODUCT

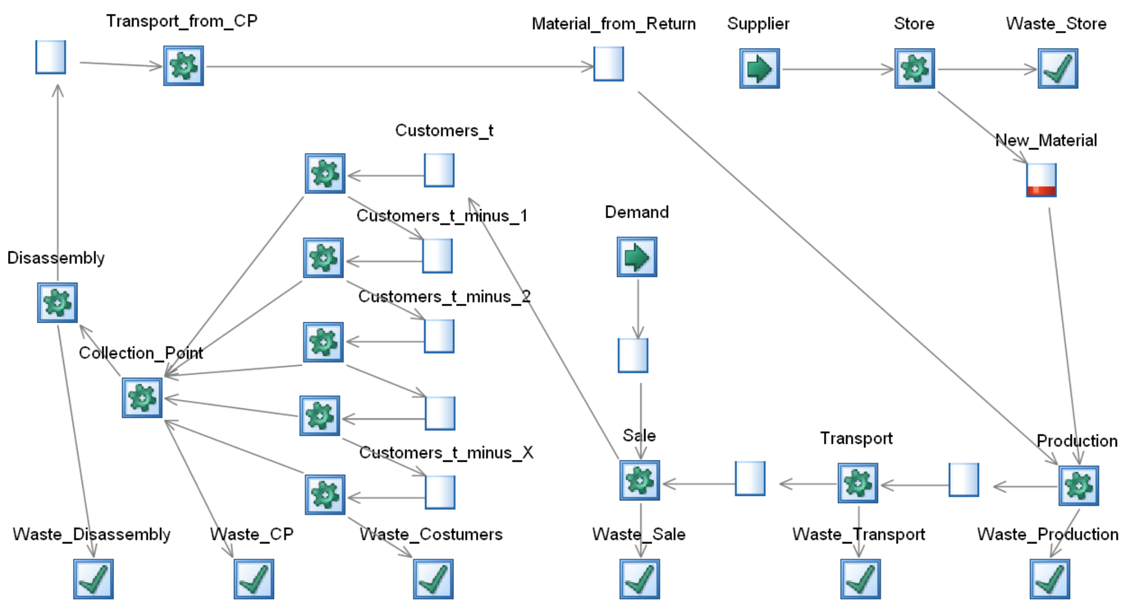
The rule how the the product is returned is determined by stochastic relationships that can describe the volume of demand for products, its rate of return and disposal factor (waste). The relationship between material flows and their rates are described in *Figure 1*, where:
d – demand for products,
α – rate at which products are returned to manufacturer,
β – rate at which returned products are used,
ω – disposal (it means the quantity of products which are not returned or reused),
$\omega = (1-\alpha)d + (1-\beta)\alpha d$

Figure 1 Relations between material flows and their rates



This kind of models are based on such relationships and they are usually different from each other in the way how the function of total costs is designed. Stochastic models for reverse approach can be also designed as static and dynamic models. They can differ in whether they are defined for a specific period or contemplate an infinite planning horizon. Also the demand influences the diversity of these models because return can depend on demand, probability or be given randomly.
Illustration when the return of product is determined by demand from longer time period like last period can be seen in *Figure 2*. This figure illustrates the principle of returning the product from several previous periods, where storage means time the product is in possession of customers. The model is designed for demonstration of the supply process with one type of material for one product and it illustrates sub-processes in remanufacturing of product.

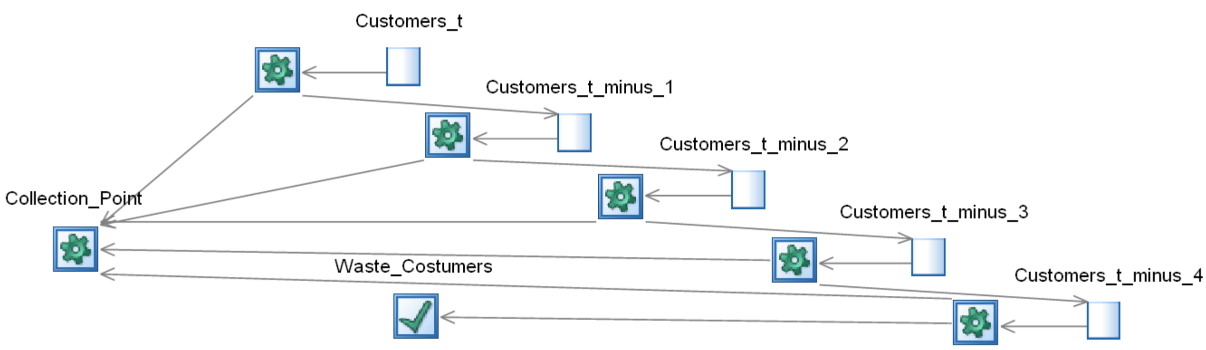
Figure 2 Model for return of product from several previous periods (Simul8)



3 SCENARIOS OF RETURN

Return of the product back to the manufacturer (Figure 3) is influenced by the probability $PR(R)_{it}$, determining the amount of products returned back to the manufacturer from the sale in the previous period t-1 ($D_{i(t\text{-}1)}$) in period t, therefore $R_{it} = PR(R)_{it} D_{i(t-1)}$, for $i = 1, 2, ...n;$ $t = 1, 2, ...T.$

Figure 3 Probability of return from 5 previous period



If the return of product doesn't depend only on sales in the previous period t–1 ($D_{it\text{-}1}$) but the product is returned from sales in several previous time periods before period t, there is a probability $PR(Rx)_{it}$. This probability represents probability of return for product i, where x represents the number of periods t, how long the product is in possession of customers, for example $PR\ (R3)_{it}$ represents the probability of product´s return in third period from the period in which the product was sold.

Parameter x is from the range 1 to *X*, therefore $x \in \langle 1; X \rangle$, where upper value of *X* is a "limit" after which it is expected that the product will not return back to the manufacturer and ends up as waste. Probability distribution of return has decreasing character because the most of the return is realized in the period immediately after the sale and then the option for return of products drastically decreases.

There are many options for calculation of the probabilities of the product´s return. Probably the simplest assumption is **the first scenario** which bases calculation of return on the amount of all sold products in the period connected with certain probability of return from that period. In this variant the sum of the probabilities is less than one and assumes that a certain percentage of products will never return to the producer and end up as waste. This percentage is determined by the difference between the sum of the probabilities and the number 1.

This determination of return can be represented as follows:

$$R_{it} = PR(R1)_{it} D_{i(t-1)} + PR(R2)_{it} D_{i(t-2)} + \ldots + PR(RX)_{it} D_{i(t-X)},$$

for $i = 1, 2, \ldots n$; $t = 1, 2, \ldots T$; X – amount the period of return.

The second scenario is based on the fact that the probability $PR(R)_{it}$ is fixed and the amount of sold products is not the same in each time period. The return is calculated only from the amount of the sold products which remain in possession of customers in certain period. This method is more difficult but more realistic because it is not calculated with the products which were returned in the previous period and provides more accurate data.

This means that the probability of return is connected only with remained amount of the sold products from *X* previous periods plus amount of the all sold products in last period. Also this variant considers the "limit" *X* after which it is expected that the product will not return back to the manufacturer and ends up as waste. Probability $PR(R)_{it}$ has to be less than one[2] which ensures that after *X* periods the product will still remain in consumers or will end as waste. This determination of return can be represented by the following calculation:

$$R_{it} = PR(R)_{it} D_{i(t-1)} + PR(R)_{it}(1 - PR(R)_{it}) D_{i(t-2)} +$$
$$+ PR(R)_{it}(1 - PR(R)_{it})^2 D_{i(t-3)} + \ldots + PR(R)(1 - PR(R)_{it})^{X-1} D_{i(t-X)},$$

for $i = 1, 2, \ldots n$; $t = 1, 2, \ldots T$; X – amount the period of return.

The third scenario is a combination of the previous two scenarios. This means that as the second variant does not calculate return from the entire amount of sold products, but only from remained amount of sold products. This scenario also assumes "limit" *X* after which it is expected that the product will not return back to the manufacturer and ends up as waste.

At the same time it uses the probability distribution of return which is degressive. This characteristic decreases slightly and is not as radical as the first option, because it assumes that the return gradually decreases with the number of periods in which the product is in possession of the costumer. However the assumption that the sum of probabilities is less than one is not considered. Probabilities of different periods may be different, and their value must be less than one.

Waste is also, as it was showed in the previous variant, determinate by "limit" *X*. This determination of return can be represented by follow calculation:

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[2] In general for this variant is better to use low values of the probability of return because this variant is suitable for products with a low but similar rate of returns for all periods of product´s return.

$$R_{it} = PR(R1)_{it} D_{i(t-1)} + PR(R2)_{it} (1 - PR(R1)_{it}) D_{i(t-2)} + PR(R3)_{it}$$
$$(1 - PR(R2)_{it})(1 - PR(R1)_{it}) D_{i(t-3)} + ... + PR(RX)(1 - PR(R(X-1))_{it})$$
$$(1 - PR(R(X-2))_{it})...(1 - PR(R(X-(X-1))_{it})_{it} D_{i(t-X)}$$

for *i = 1, 2, ...n; t = 1, 2, ...T; X* – amount the period of return.

4 MODEL

Integration of the return from several previous periods in the models is important when tracking the flow of returned products in the general mathematical programming model for reverse logistics. Choosing a suitable scenario of products return is dependent on the application of the model. The first variant is suitable for products that have high rate of products return in first period following the sale and the majority of return occurs immediately after the sale. The second variant is suitable for the opposite case where rate of products return in first period after the sale is lower but more uniform throughout the periods of return.

The costs associated with calculation of product´s return by one of these scenarios can be:

CCR_{itu} - cost of collecting returned products *i* (*i* = 1, 2,... *n*) in the period *t* (*t* = 1, 2,... *T*),

CTR_{itu} - cost of transport returned products *i* (*i* = 1, 2,... *n*) in the period *t* (*t* = 1, 2,... *T*) and

CDR_{itu} - cost related to return and dismantling returned products *i* (*i* = 1, 2,... *n*) in the period *t* (*t* = 1, 2,... *T*).

The formulation of the cost function for model with the return of the product via mathematical programming can be:

$$\min \sum_{i=1}^{n}\sum_{t=1}^{T} CCR_{it} R_{it} + \sum_{i=1}^{n}\sum_{t=1}^{T} CTR_{it} R_{it} + \sum_{i=1}^{n}\sum_{t=1}^{T} CDR_{it} R_{it}$$

The third scenario for calculation of products return represent a compromise between the previous two scenarios. Contents for model with the return of the product via mathematical programming can be in this form:

$$R_{itu} = PR(R1)_{it} D_{i(t-1)} + PR(R2)_{it} (1 - PR(R1)_{it}) D_{i(t-2)} + PR(R3)_{it}$$
$$(1 - PR(R2)_{it})(1 - PR(R1)_{it}) D_{i(t-3)} + ... + PR(RX)(1 - PR(R(X-1))_{it})$$
$$(1 - PR(R(X-2))_{it})...(1 - PR(R(X-(X-1))_{it})_{it} D_{i(t-X)}$$

for *i = 1, 2, ...n; t = 1, 2, ...T; X* – amount the period of return.t

CONCLUSION

This approach allows modeling different variants of products return back to manufacturer in the general mathematical programming model for reverse logistics and also allows different variants form simulation this model. It extends classic approach to modeling of production processes because it includes the modeling of the reverse material flows and also stochastic elements in relationships among demand for products, rate of product´s return and disposal factor.

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HOW TO DETERMINE AND CONSIDER THE WEIGHTS OF INTERACTING CRITERIA IN MCDM

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Abstract

This paper introduces some possible solutions for determining and considering the weights of both, independent and interacting criteria in the multi-criteria decision-making problems. Special attention is given to the use of the methods for establishing the judgments on criteria's importance, based on the interval scale. To consider interactions among criteria, in synthesis, we firstly delineate how to complete the additive model into the multiplicative one with synergic and redundancy elements. Furthermore, we present the concept of the fuzzy Choquet integral, adapted to the multi-attribute value theory. Making judgments about the criteria's importance and different approaches of obtaining the aggregate alternatives' values are illustrated by a real-life case for the selection of the most appropriate storage array.

Keywords: *information technology, multi-criteria decision-making, synergy*

JEL Classification: C81, M15
AMS Classification: 90B50

1 INTRODUCTION

In the frame procedure for multi-criteria decision-making (MCDM) by using the group of methods, based on assigning weights, the problems are approached step-by-step [2]: *Problem definition, Elimination of unacceptable alternatives, Problem structuring, Measuring local alternatives' values, Criteria weighting, Synthesis, Ranking,* and *Sensitivity analysis*. In this paper, special attention is given to criteria weighting and to synthesis. Since, in practical applications, decision makers very often tell the relative importance of criteria directly with difficulty, we describe the use of the methods for establishing the judgments on criteria's importance, based on the interval scale.

Then we introduce some possible solutions for considering the weights of independent and interacting criteria in the MCDM problems. Interactions among criteria should be considered in measuring the global phenomena like globalization, sustainable development and (corporate) social responsibility, as well as in solving other complex problems like the house purchase decision [4], and in the selection of several types of information infrastructure. An overview of the commonly used leading MCDM literature (e.g. [1], [4]) can let us report that decision-analysis theorists and practitioners tend to avoid interactions by constructing independent (or supposed to be so) criteria. In synthesis, the additive model is used where the mutual preferential independence of criteria is assumed [4]. However, the synthesis by the additive model may hide synergies and redundancies. Therefore we firstly present how to complete the additive model into the multiplicative one with synergic and redundancy elements. Furthermore, we present the concept of the fuzzy Choquet integral [5], adapted to the multi-attribute value theory, to obtain the aggregate alternatives' values.

Making judgments about the criteria's importance and different approaches of obtaining the aggregate alternatives' values are illustrated by a real-life case for the selection of the most appropriate storage array in the information technology (IT) company that wants to present possible solutions to their current and potential customers.

2 ESTABLISHING THE CRITERIA'S IMPORTANCE

To help decision makers express the judgments about the criteria's importance, the methods based on ordinal (e.g. SMARTER), interval (e.g. SWING and SMART) and the ratio scale (i.e. AHP) [1] can be used besides the direct weighting.

Let us describe how to use the methods for establishing the judgements on criteria's importance, based on the interval scale. In SMART, a decision maker is first asked to assign 10 points to the least important attribute change from the worst criterion level to its best level, and then to give points (≥ 10, but ≤ 100) to reflect the importance of the attribute change from the worst criterion level to the best level relative to the least important attribute range [6]. In SWING, a decision maker is asked first to assign 100 points to the most important attribute change from the worst criterion level to the best level, and then to assign points (≤ 100, but ≥ 10) to reflect the importance of the attribute change from the worst criterion level to the best level relative to the most important attribute change [6].

3 CONSIDERING INTERACTIONS AMONG CRITERIA

In the Multi-Attribute Value (or Utility) Theory and the methodologies that were developed on its bases (e.g. Simple Multi Attribute Rating Approach – SMART, the Analytic Hierarchy Process – AHP [1]), the additive model is usually used when obtaining the aggregate alternatives' values in synthesis as the sum of the products of weights by corresponding local alternatives' values. When the criteria are structured in one level only, the aggregate alternatives' values are obtained by:

$$v(X_i) = \sum_{j=1}^{m} w_j v_j(X_i), \quad \text{for each } i = 1, 2, \ldots, n, \tag{1}$$

where $v(X_i)$ is the value of the i^{th} alternative, w_j is the weight of the j^{th} criterion and $v_j(X_i)$ is the local value of the i^{th} alternative with respect to the j^{th} criterion. When the criteria are structured in two levels, the aggregate alternatives' values are obtained by:

$$v(X_i) = \sum_{j=1}^{m} w_j \left(\sum_{s=1}^{p_j} w_{js} v_{js}(X_i) \right), \quad \text{for each } i = 1, 2, \ldots, n, \tag{2}$$

where p_j is the number of the j^{th} criterion sub-criteria, w_{js} is the weight of the s^{th} attribute of the j^{th} criterion and $v_{js}(X_i)$ is the local value of the i^{th} alternative with respect to the s^{th} attribute of the j^{th} criterion.

The use of the additive model written by (1) and (2) is not appropriate when there is an interaction among the criteria. According to Goodwin and Wright [4], the most well known of the models which can handle the interactions among the criteria that express redundancy and synergy is the multiplicative model. Let us suppose that the MCDM problem is being solved with respect to two criteria only – this simplification is made to explain the bases of multiplicative models. As proposed in [4], the value of the i^{th} alternative $v(X_i)$ is:

$$v(X_i) = w_1 v_1(X_i) + w_2 v_2(X_i) + w_{1,2} v_1(X_i) v_2(X_i), \quad \text{for each } i = 1, 2, \ldots, n, \tag{3}$$

where w_1 is the weight of the first and w_2 is the weight of the second criterion, $v_1(X_i)$ is the local value of the i^{th} alternative with respect to the first and $v_2(X_i)$ is the local value of the i^{th} alternative with respect to the second criterion. The last expression in the above sum (3), which involves multiplying the local alternatives' values and the weight of the synergy between the first and the second criterion $w_{1,2}$, represents the interaction between the first and the second criterion that expresses the synergy between these criteria. To consider a negative interaction (redundancy) between the criteria, the product of the local alternatives' values and the weight of the interaction between the first and the second criterion can be deducted from the sum obtained with the additive model. [3]

Further, the concept of fuzzy measure has been introduced to consider interactions among criteria [9]: in order to have a flexible representation of complex interaction phenomena between criteria, it is useful to substitute to the weight vector *w* a non-additive set function on *K* – the set of criteria – allowing to define a weight not only on each criterion, but also on each subset of criteria. A suitable aggregation operator, which generalizes the weighted arithmetic mean, is the discrete Choquet integral. Following [5] and [9], this integral is viewed here as an *m*-variable aggregation function; let us adopt a function-like notation instead of the usual integral form, where the integrand is a set of *m* real values, denoted by $v = (v_1, \ldots, v_m) \in \Re^n$. The (discrete) Choquet integral of $v \in \Re^n$ with respect to *w* is defined by:

$$C_w(v) = \sum_{j=1}^{m} v_{(j)} \left[w(K_{(j)}) - w(K_{(j+1)}) \right], \tag{4}$$

where $(\cdot)$ is a permutation on *K*, such that $v_{(1)} \leq \ldots \leq v_{(m)}$. Also, $K_{(j)} = \{(j), \ldots, (m)\}$.

4 A REAL-LIFE CASE

The MCDM model for the selection of the most suitable storage array was built together with an IT company with the aim of presenting possible solutions to their current and potential customers: medium-sized companies. The frame procedure for MCDM, based on assigning weights, was followed. The storrage arrays that can be offered to medium-sized companies are described as alternatives: Sun Storage 6580 Array (Alternative 1) [11], HP EVA 4400 (Alternative 2) [7], IBM Storwize V7000 Unified Disk System (Alternative 3) [8] and E7900 Storage System (Alternative 4) [10]. The criteria hierarchy includes the 'costs' (Purchase price, Space, Base unit power), 'capacity' (Base unit capacity, Maximal capacity, Host connectivity), and 'quality' attributes (Management, Additional features, Security features).

On the bases of experiences and detailed data from the principal, engineers in the considered IT company responsible for pre-sales support expressed their judgments about the criteria's importance. The first level criteria weights were determined directly. The importance of the capacity attributes was assessed with the SWING method. 100 points were given to the change from the worst to the best base unit capacity, which is considered the most important criterion change. With respect to this change importance, 80 points were given to the change from the lowest to the highest maximal capacity, and 20 points were given to the change from the worst to the best host connectivity level. The importance of the quality attributes were assessed with the SMART method. 10 points were given to the change from the worst to the best management, which is considered the least important criterion change. With respect to this, 30 points were given to the change from the lowest to the highest additional features, and 60 points to the change from the worst do the best security features. The importance of the costs attributes was assessed by using the SMART method, as well. The weights of the attributes – second level criteria, and the first level criteria are presented in Table 1, while measuring local alternatives' values is not given a special attention in this paper.

Table 1 The criteria structure and the weights for the selection of storage arrays.

First level criteria	Weights of the first level criteria	Second level criteria	Weights of the second level criteria
Costs	$w_1 = 0.25$	Purchase price	$w_{11} = 0.6$
		Space	$w_{12} = 0.3$
		Base unit power	$w_{13} = 0.1$
Capacity	$w_2 = 0.4$	Base unit capacity	$w_{21} = 0.5$
		Maximal capacity	$w_{22} = 0.4$
		Host connectivity	$w_{23} = 0.1$
Quality	$w_3 = 0.35$	Management	$w_{31} = 0.1$
		Additional features	$w_{32} = 0.3$
		Security features	$w_{33} = 0.6$

The alternatives' values with respect to the first level criteria and the aggregate alternatives' values, obtained by (2) in synthesis, are shown in Table 2. On the bases of experiences and detailed data from the principal, engineers in the considered IT company responsible for pre-sales support evaluated that there is synergy between 'capacity' and 'quality'. In the concept of the multiplicative model, this means that $w_2 + w_3 + w_{2,3} > 0{,}75$ (Table 1); they evaluated that $w_2 + w_3 + w_{2,3} = 0{,}9$, $w_{2,3} = 0{,}15$. They also evaluated that there is synergy between 'costs' and 'quality': $w_1 + w_3 + w_{1,3} > 0{,}6$ (Table 1), $w_1 + w_3 + w_{1,3} = 0{,}7$, $w_{1,3} = 0{,}1$, and redundancy between 'costs' and 'capacity': $w_1 + w_2 + w_{1,2} < 0{,}65$ (Table 1), $w_1 + w_2 + w_{1,2} = 0{,}55$, $w_{1,2} = -0{,}1$. Table 2 presents the aggregate alternatives' values, obtained by considering synergic and redundancy elements with the additive model, completed into multiplicative one:

$$v(X_i) = v(X_i)_A + w_{1,3}v_1(X_i)v_3(X_i) + w_{2,3}v_2(X_i)v_3(X_i) - w_{1,2}v_1(X_i)v_2(X_i), \qquad (5)$$

where $v(X_i)_A$ is the aggregate value of the i^{th} alternative, obtained by the additive model.

In the concept of the Choquet integral, the evaluated synergy between 'capacity' and 'quality' means: $w_{2,3} > w_2 + w_3$; $w_2 + w_3 = 0{,}75$ (Table 1), and $w_{2,3} = 0{,}90$. Synergy between 'costs' and 'quality' means: $w_{1,3} > w_1 + w_3$; $w_1 + w_3 = 0{,}6$, and $w_{1,3} = 0{,}7$. The evaluated negative synergy between 'costs' and 'capacity' means: $w_{1,2} < w_1 + w_2$; $w_1 + w_2 = 0{,}65$, $w_{1,2} = 0{,}55$. Table 2 presents the Choquet integrals, obtained by (4). For instance, for Alternative 4, where $v_3 < v_1 < v_2$ (see Table 2), we have:

$$C_w(v_1, v_2, v_3) = v_3[w_{3,1,2} - w_{1,2}] + v_1[w_{1,2} - w_2] + v_2w_2, \qquad (6)$$

where $w_{3,1,2} = 1$. Following (4), the Choquet integral for other alternatives was expressed.

Table 2 The alternatives' values.

	Alternative 1	Alternative 2	Alternative 3	Alternative 4
Value with respect to 'costs' v_1	0.431	0.415	0.679	0.516
Value with respect to 'capacity' v_2	0.178	0.187	0.534	0.980
Value with respect to 'quality' v_3	0.220	0.450	0.430	0.450
Aggregate alternative's value v – additive model	0.256	0.336	0.534	0.678
Aggregate alternative's value v – multiplicative model	0.264	0.360	0.561	0.717
Choquet integral C	0.260	0.359	0.523	0.672

5 CONCLUSIONS

When using the Choquet integral approach, its main disadvantage comes into front. Due to the ranking of the alternative's values with respect to the criterion on the observed level, some of the synergies and redundancies might not be considered. In the above presented real-life case it was not possible to include the synergy between 'capacity' and 'quality' in (4). Moreover, the ranking of the alternative's values can differ for each of the considered alternatives, which results in adapting (4) to each alternative. On the other hand, when using the additive model, completed into the multiplicative one, the ranking of the alternative's value with respect to the criterion on the observed level is not needed. Once the weights of the synergy and redundancy between criteria are determined, decision makers can use the same formula for all alternatives. The above described simplification regarding the consideration of redundancies in the additive model, completed into the multiplicative one, might allow decision makers to use the most preferred computer supported multi-criteria methods, based on the additive model, to obtain the aggregate values, improved for positive and negative interactions between criteria.

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VALUE AT RISK IN LIGHT OF CRISIS

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Abstract
Value at Risk (VAR) represents statistical tool that measures an entity's exposure to market risk. Aim of this article is to discuss the appropriateness of the VAR measure application in the turbulent times of crises. Concept of the VAR and volatility estimation will be described in detail and applied for selected portfolio of stocks.

Keywords: *Value at Risk, Market Risks, Volatility, Crisis*

JEL Classification: C44
AMS Classification: 90C15

Introduction
The current business environment due to the lingering economic crisis is very complex and unstable. Business entities operating in the financial sector are therefore exposed to an uncertain future with the profit or uncertain loss. In this case, not only risking your capital but also the resources of clients. For this reason risk assessment in all major financial institutions represents very importatn process. Under the evaluation of the risks we understand the quantification of present future risk, which stems from the active participation of institutions in the financial markets. The experience forced investors to seek a tool to measure risk exposure, which led to the expansion of VAR applications (Value at Risk). VAR represents a method of assessing risks that uses standard statistical techniques commonly used in other professional sectors. However, there is no precise definition of VAR, we can find several definitions in literature such as:
P. Jorion, *"VaR summarizes the worst loss for a specified time horizon at a given confidence interval."*
C. Butler*: "VaR measures the worst expected loss that the institution faces during a given time interval under normal market conditions at a given level of significance. VaR assess this risk, using statistical and simulation models which are developed in order to determine the volatility of the portfolio of the bank. "*
In comparison with traditional tools for risk measurement, VAR provides a comprehensive view of the risk portfolio, which includes leverage, correlations and current market position. Resulting in the VAR is a real prospective rate. VAR is used not only to analyze financial derivatives, but all financial instruments. Moreover, it is possible to extend the VAR methodology in addition to market risk for other types of financial risks.

1 THE CALCULATION OF VAR

VAR represents a value of the worst possible loss over a given (chosen) horizon above which the real loss is defined by the probability (confidence) level. This definition in itself contains two quantitative factors, time horizon and confidence level that are chosen by user.
Let c is chosen confidence level and L is a real loss, measured as a positive value (VAR is also shown as a positive number). In general, VAR is the lowest loss in absolute value, such that

$$P(L > VAR) \leq 1 - c \quad (1)$$

As an example we can give the chosen 99 percent confidence level or c = 0.99. Then VAR represents a level of loss to which the probability of overrun is less than one percent.

1.1 The steps of calculating the VAR

In calculation of the VAR is necessary to carry out the following steps:

- to determine the current market value of portfolio,
- to calculate the degree of variability in risk,
- to determine the time horizon,
- determining confidence levels,
- to calculate the worst possible loss (level VAR).

With such a simplified procedure of calculation the idea of VAR may seem clear and understandable, but the problem of estimating the empirical probability distribution of income is still current and is not resolved satisfactorily. There are several basic methods of calculating VAR, which differ in the assumptions of the model and the way how to estimate the distribution function of income. Based on assumptions about the shape of the distribution of income the VAR is divided into: parametric VAR and nonparametric VAR. In this paper we will focus on parametric approach and thus in next section this approach will be shortly discussed.

1.2 Parametric VAR measure

The calculation of parametric VAR represent simplification of market risks analysis by applying the assumption about the distribution of returns. If it is a given case, the VAR can be calculated directly from the standard deviation (σ) using a multiplier dependent on the chosen confidence level. This approach is called *parametric* because the calculation of VAR estimates uses parameters such as standard deviation instead of simply reading quantiles of the distribution itself .

Suppose we decide to use normal distribution for data description. First we need to transform, the general distribution $f(w)$ into a standard normal distribution $\varphi(\epsilon)$ where ϵ has zero mean and its standard deviation is is equal to 1, N (0, 1). To connect worst expected value of the portfolio W^* with income rate R^* we use the following formula: $W^* = W_0(1+R^*)$. In general, when measuring the value at risk (VAR) R^* reaches negative values and can therefore be written as $-|R^*|$. Moreover, it is possible to link R^* from known data with standard deviations of income rate (σ), the expected rate of income (μ) and the tabulated values of the normal distribution for the chosen confidence level (α) using the following equation:

$$-\alpha = -\frac{|R^*| - \mu}{\sigma} \qquad (2)$$

Now using α we can find marginal value of income R^* and VAR rate. Modifying equation (2) we get

$$R^* = -\alpha\sigma + \mu \qquad (3)$$

In order to simplify mentioned example we assume that the parameters μ and α are measured on an daily basis. The time period that we consider is Δt expressed in years. To compute VAR we can apply following form

$$VAR = E(W) - W^* = W_0(R^* - \mu) = W_0 \alpha\sigma\sqrt{\Delta t} \qquad (4)$$

In other words, the value of VAR can be simply calculated by multiplying the standard deviation of income rate (σ) and modifying the parameter (α), which is directly linked to a confidence level and time horizon.

To compute parametric VAR measure it is important to estimate the variance of returns of certain asset. In our paper we focused on two traditional parametric time-series modeling approaches to estimate market risk: moving averages and GARCH[1] estimation. One of very widely used methods for volatility estimation is moving averages of fixed length. Usually employed length of the time window is 20 (applied in this paper) or 60 trading days (approx. one month or quarter). To construct the estimation of the volatility we assumed that we observe returns r, over M (20) days and then we have

[1] Generalized autoregressive conditional heteroskedastic

$$\sigma_t^2 = (1/M)\sum_{i=1}^{M} r_{t-i}^2 \qquad (5)$$

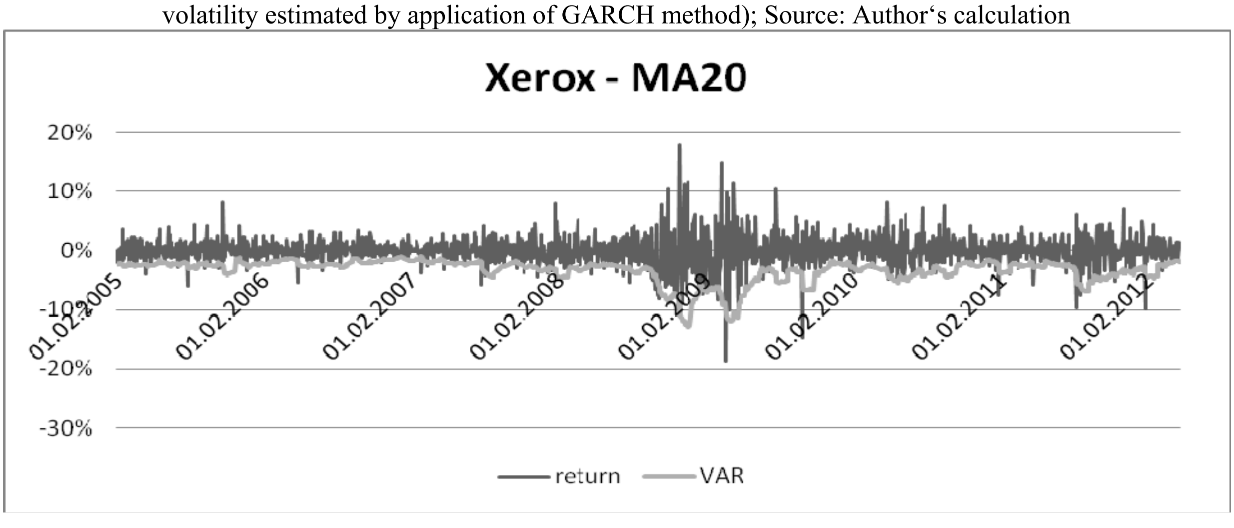
By use of this formula we obtained volatility estimates for given time period. This approach has a few drawbacks and we will later discuss their effects on the obtained estimates. Main concern is connected with fact that recent information receives the same weight as older observations in the time window that maybe no longer relevant.

This was one of the reasons why volatility estimation from time series moved to models that put more weight on recent information. First model of such nature was ARCH model presented by Engle (1982) developed to GARCH by Bollerslev (1986). This model assumes that the volatility of returns follows a predictable process. Conditional volatility depends on the latest information but also on the previous volatility. Let h_t be conditional volatility defined by using information up to time *t-1* and previous day's return r_{t-1}. To estimate volatility in our research we used simple model defined as GARCH(1,1) process defined as follow

$$h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1} \qquad (6)$$

Obtained volatility estimates were used to generate historical simulation of VAR measure and then we compared them to daily returns of three different stocks[2]: Dell, Hewlett Packard (HP) and Xerox to see how well VAR predicted market risks with emphasis on the turbulent crisis periods. The results have shown that if moving average (MA20) method is used to estimate volatility even in the times of relatively stable market condition VAR estimates are underestimating the real market risks. This is caused mainly by the fact that weights assigned to information about returns in the moving window are the same. If we take a detail look on the performance of moving average estimates based 95% VAR then we can see that in all cases in years 2008 and 2011 in more than 5% of days real losses were higher than estimated value of VAR. In case of 95% VAR estimates based on GARCH volatility estimates real losses were higher than estimated VAR in 2008 for all cases and in two cases in 2011 and once in 2010. Real losses for Xerox were higher than 95% VAR estimates based on MA20 volatility estimates in years 2006 (5.2% cases), 2007 (5.6%), 2008 (8.3%) and 2011 (7.5%) and in case based on GARCH volatility estimates in years in which the markets were mostly influenced by crisis (2008 (5.9%) and 2011 (6.7%)). As we can see in Figure 1 most volatility for price of Xerox stocks occurred at the end of year 2008 and beginning of year 2009 and also during the year 2011 after period of relatively stable variance sharp increase in volatility can be observed.

Figure 1 95% VAR compared to real data for Xerox (MA20 – moving average 20 days window, GARCH – volatility estimated by application of GARCH method); Source: Author's calculation



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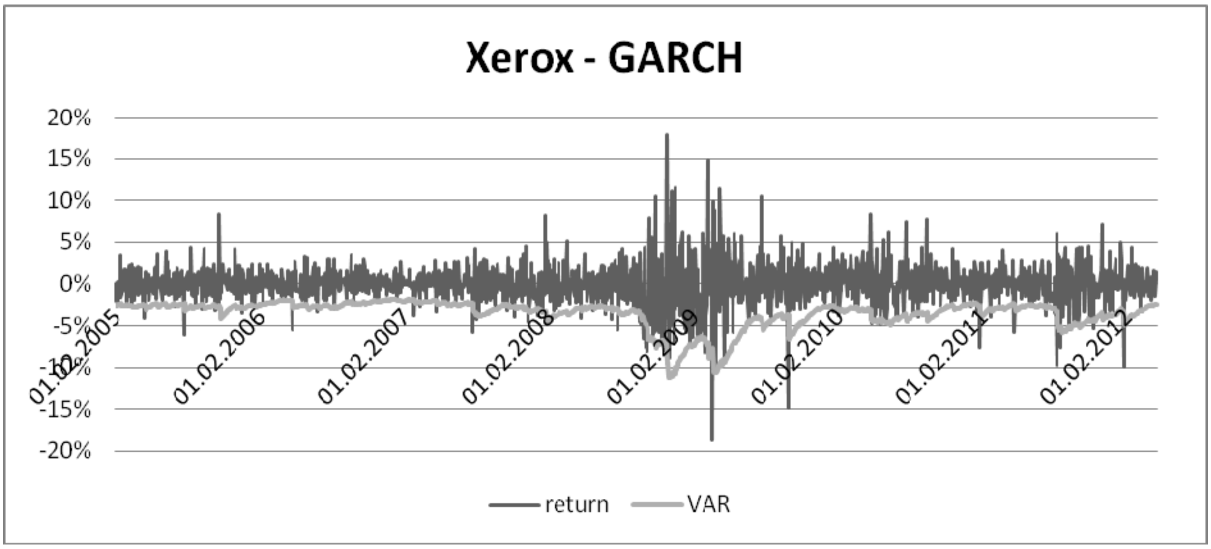
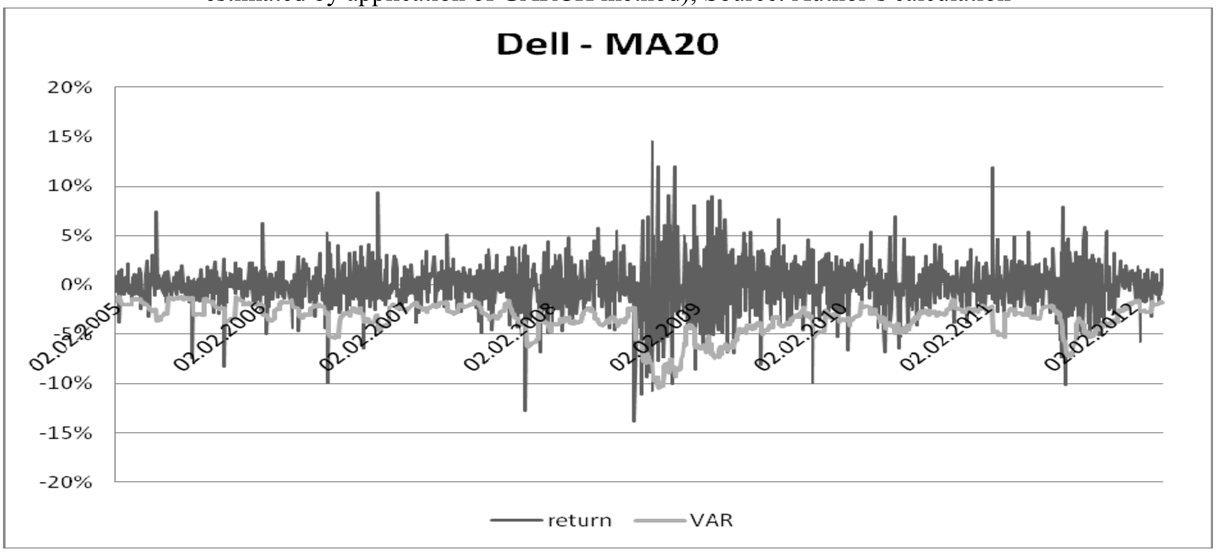
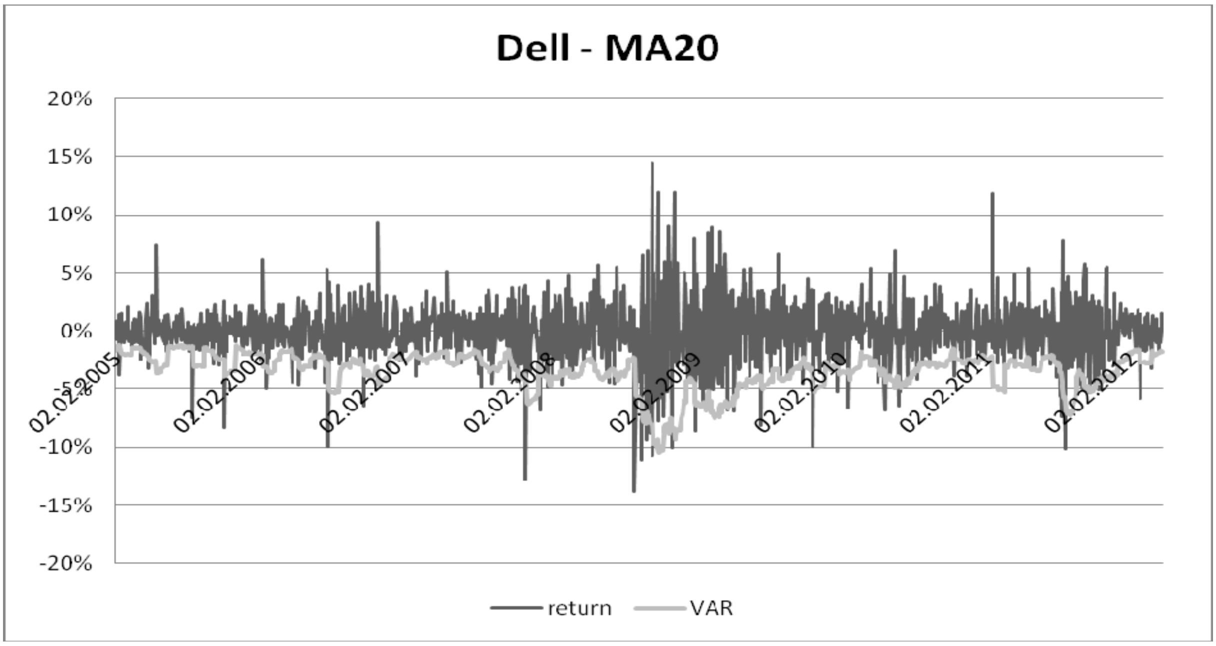
[2] Daily stock prices data were obtained from finance.yahoo.com for time period starting from 3rd of January 2005 till 17th of April 2012



Real losses for Dell were higher than 95% VAR estimates based on MA20 volatility estimates in years 2005 (6.5%), 2007 (7.2%), 2008 (7.1%), 2010 (6.7%) and 2011 (6.0%) and in case based on GARCH volatility estimates in years in which the markets were mostly influenced by crisis (2008 (7.1%) and 2010 (5.6%)). As it can be seen from Figure 2 most significant changes in variance of stock prices for Dell occurred in years 2005, 2008, 2010 and at the end of the year 2011.

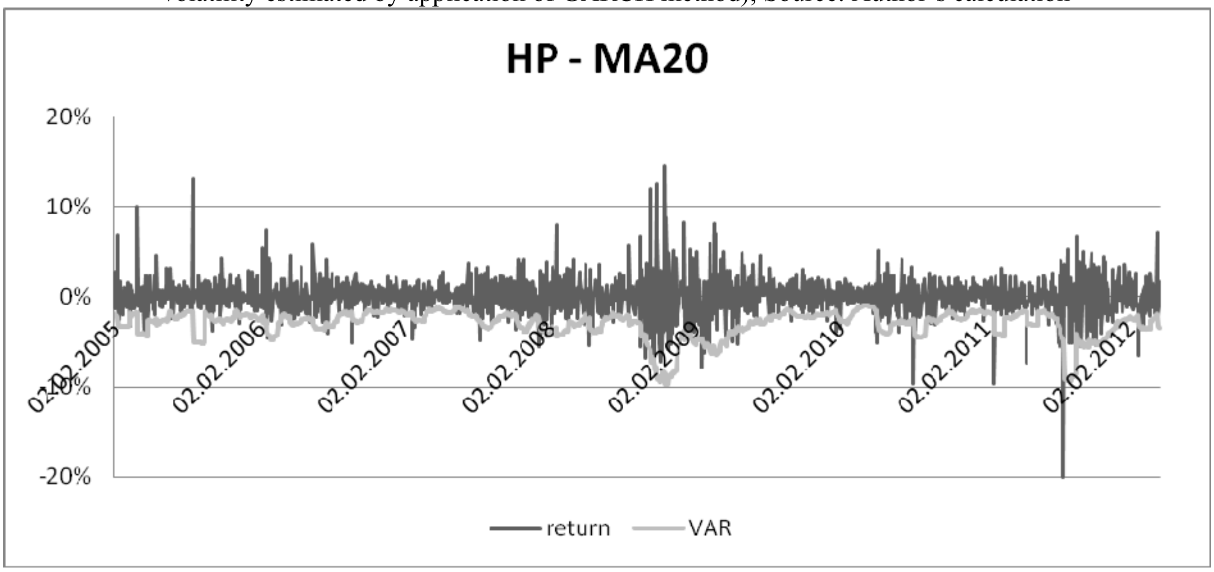
Figure 2: 95% VAR compared to real data for Dell (MA20 – moving average 20 days window, GARCH – volatility estimated by application of GARCH method); Source: Author‘s calculation

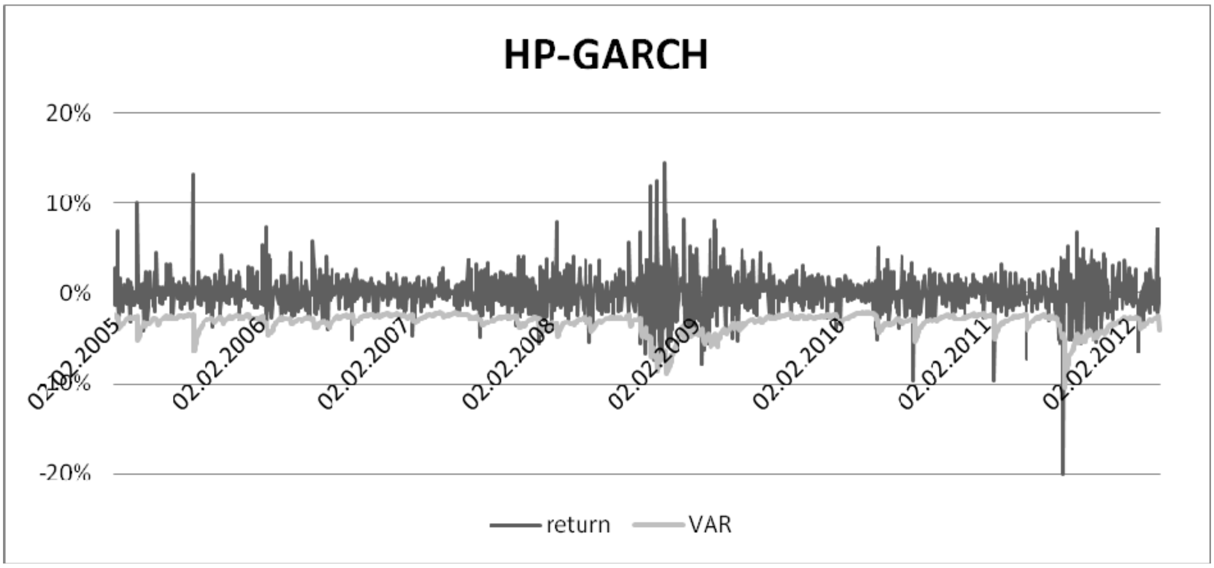




Only in case of HP suitability of both methods was very close in terms of occurrence of real losses being higher than 95% VAR (more than 5% of days estimated VAR was exceeded for MA20 in 2008 (5.5%), 2010 (6.7%) and 2011 (5.6%) and for GARCH in 2008 (5.5%) and 2011 (6.3%)). As it can be seen from Figure 3 most significant changes in variance of stock prices for HP occurred in years 2005, 2008, beginning of year 2009 and during the year 2011.

Figure 3 95% VAR compared to real data for Hewlett Packard (MA20 – moving average 20 days window, GARCH – volatility estimated by application of GARCH method); Source: Author‘s calculation





2 CONCLUSIONS

VAR represents a relatively simple tool for measuring the market risks. In this paper we focused on performance of VAR measure applied for returns of three different stocks of IT companies (Xerox, Dell, HP) during the years 2005 – 2012. To examine the performance of VAR we calculated parametric VAR based on volatility estimates calculated by use of moving averages and GARCH approach. Estimated 95% VAR were compared with real losses during above mentioned period. Results approved the expected fact that VAR based on moving averages variance estimates produce less reliable results because same weights are applied for all historical information. Moving averages based VAR do not fullfill expectations even in the years of relativly stable market conditions. In most of cases GARCH based VAR produced more suitable information about market risks and only scarcely moving average based VAR outperformed the GARCH based VAR. Eventhough in the most turbulent periods of financial market conditions GARCH based VAR measure was not able to predict future market risks based only on the historical information about given stock returns and thus would be enhanced by employing additional risks factors explaining market movements.

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EVALUATION OF THE DIRECT AND MULTI-STOP FREQUENCY BASED HEURISTICS FOR THE INVENTORY ROUTING PROBLEM

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Abstract
In this paper we observe a multi-product multi-period Inventory Routing Problem (IRP) in fuel delivery with homogeneous multi-compartment vehicles and deterministic consumption that varies over each petrol station and each fuel type. The IRP total costs include routing costs that are directly dependent on the vehicle travel distances, and inventory costs that are directly dependent on the average daily inventory levels. We present a Mixed Integer Linear Programming (MILP) model and two heuristics approaches which are based on the direct delivery and the multi-stop delivery with the inter-shipment frequencies. The main intention of our research is to evaluate these two commonly used heuristics for solving the IRP in fuel delivery.

Keywords: *Inventory Routing Problem, Heuristics, Direct delivery, Multi-stop delivery*

JEL Classification: C61
AMS Classification: 90C11, 90B05, 90B06, 90C59

1 INTRODUCTION

There are two sub-problems in fuel delivery, transportation of different fuel types in one-to-many distribution system and the impact of replenishment quantities on the inventory levels at petrol stations. The Inventory Routing Problem (IRP) tries to solve those two sub-problems simultaneously with the objective of minimizing the total costs which include vehicle routing and inventory carrying costs. IRP solution determines the quantity of the goods and the time of delivery as well as the vehicle routing. VMI (Vendor Managed Inventory) concept is a prerequisite for the implementation of the IRP; in which supplier determines the order quantity and the time of delivery. In the secondary fuel distribution, different fuel types are transported from one depot location to a set of petrol stations by a designated fleet of multi-compartment vehicles, and for which a single oil company has the control over all of the managerial decisions over all of the resources. As a consequence, the full VMI concept can be applied. In this paper we observe a multi-product multi-period IRP in fuel delivery with homogeneous multi-compartment vehicles and deterministic consumption that varies over each petrol station and each fuel type. The main objective is to minimize IRP total costs that include routing costs which are directly dependent on vehicle traveled distances, and the inventory costs which are directly dependent on the average daily inventory levels. We present a Mixed Integer Linear Programming (MILP) model and two heuristics approaches which are based on the direct delivery and the multi-stop delivery with the inter-shipment frequencies. The main intention of our research is to evaluate two commonly used heuristics for solving the IRP in fuel delivery. For a detailed review on the IRP we refer the reader to Moin and Salhi (2007) and Andersson et al. (2010).

This paper is organized as follows: The problem formulation is given in Section 2. Section 3 presents a mathematical formulation. A description of the proposed heuristic models is given in Section 4, and computational results are presented in Section 5. Finally, conclusions are given in Section 6 as well as directions for further research.

2 PROBLEM FORMULATION

In this section we describe distribution structure of the observed IRP in fuel delivery. The delivery quantities of $\boldsymbol{J}$ fuel types for a given set of $\boldsymbol{I}$ petrol stations must be determined for each day through the entire planning horizon $\boldsymbol{T}$. Fuel is transported by homogeneous multi-compartment unlimited vehicle fleet. The total number of compartments is noted as $\boldsymbol{K}$. Only full compartments are delivered to petrol stations and vehicles must be fully loaded when leaving the depot. The total amount of delivered fuel allocated to a single fuel type is noted as $\boldsymbol{d_k}$ where $\boldsymbol{k} \in \{1, 2, ..., \boldsymbol{K}\}$. Every petrol station $\boldsymbol{i}$ has a constant consumption $\boldsymbol{q_{ij}}$ of each fuel type $\boldsymbol{j}$, and the intensity of the consumption varies over different stations and different fuel types. Petrol stations are equipped with underground tanks of known capacity $\boldsymbol{Q_{ij}}$ (one for each fuel type). Stations can be served only once during the day (the observed time period). It is not allowed that inventory levels in petrol stations for any fuel type fall below their defined fuel consumption $\boldsymbol{q_{ij}}$. The total inventory costs are assumed to be dependent on the sum of the average stock levels in each day of the planning horizon, whereas transport costs depend on a vehicle's travel distance. One vehicle can visit up to three stations per route.

3 MATHEMATICAL FORMULATION

The proposed mathematical formulation can be described as the MILP model. Overall, the objective of the proposed MILP model is to minimize the total costs (***IC***+***RC***). The inventory segment (2) of the objective function attempts to minimize the total inventory costs for the fuel stored in all of the petrol stations during the observed planning horizon. Those costs are based on the average daily inventory levels at the petrol stations. The routing segment (3) of the objective function attempts to minimize the total travel distance of all of the routes that are used for delivery during the observed planning horizon by solving a set of assignment problems. The routing costs are calculated as a sum of all of the travel costs incurred by deliveries in the observed planning horizon.

$$x_{ijtk} = \begin{cases} 1 \text{ - if petrol station } i \text{ is supplied with fuel type } j \text{ in time period } t \text{ with } k \text{ compartments} \\ 0 \text{ - otherwise} \end{cases}$$

$$y_{pqwt} = \begin{cases} 1 \text{ - if petrol stations } p, q, \text{and } w \text{ are supplied in the same route in time period } t \\ 0 \text{ - otherwise} \end{cases}$$

$$y_{pqt} = \begin{cases} 1 \text{ - if petrol stations } p \text{ and } q \text{ are supplied in the same route in time period } t \\ 0 \text{ - otherwise} \end{cases}$$

$$y_{pt} = \begin{cases} 1 \text{ - if petrol station } p \text{ in supplied with direct delivery in time period } t \\ 0 \text{ - otherwise} \end{cases}$$

$$H_{it} = \begin{cases} 1 \text{ - if petrol station } i \text{ is supplied in time period } t \\ 0 \text{ - otherwise} \end{cases}$$

Indices:

$\boldsymbol{i,p,q,w}$ - petrol stations ($\boldsymbol{i, p,q,w} \in \{1, 2,\ldots, \boldsymbol{I}\}$,)
$\boldsymbol{j}$ - fuel types ($\boldsymbol{j} \in \{1, 2,\ldots, \boldsymbol{J}\}$)
$\boldsymbol{t, z}$ - time period or day in the planning horizon $\boldsymbol{T}$ ($\boldsymbol{t, z} \in \{1, 2,\ldots, \boldsymbol{T}\}$)
$\boldsymbol{k}$ - number of compartments ($\boldsymbol{k} \in \{1, 2,\ldots, \boldsymbol{K}\}$)
$\boldsymbol{d_k}$ - delivery quantities ($\boldsymbol{k} \in \{1, 2,\ldots, \boldsymbol{K}\}$) that correspond to the total amount of delivered fuel

Parameters:

$\boldsymbol{S^0_{ij}}$ - stock level of fuel type $\boldsymbol{j}$ at station $\boldsymbol{i}$ at the beginning of the planning horizon
$\boldsymbol{q_{ij}}$ - consumption of the fuel type $\boldsymbol{j}$ at station $\boldsymbol{i}$
$\boldsymbol{c_{inv}}$ - inventory carrying costs per day
$\boldsymbol{c_r}$ - transportation costs per unit of traveled distance

Q_{ij} - capacity of the underground reservoir for the fuel type ***j*** at station ***i***
r_{pqw} - minimum length of route that includes petrol stations ***p***, ***q***, and ***e***
r_{pq} - minimum length of route that includes petrol stations ***p*** and ***e***
r_{p} - minimum length of route that includes only petrol station ***p***

Objective function:

$$\text{Minimize} \rightarrow IC + RC \tag{1}$$

$$IC = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T}\left(\left(S^0{}_{ij} - t \cdot q_{ij} + \frac{q_{ij}}{2}\right) + \sum_{z=1}^{t}\sum_{k=1}^{K} x_{ijzk} \cdot d_k\right) \cdot c_{inv} \tag{2}$$

$$RC = \sum_{t=1}^{T}\sum_{p=1}^{I}\left(y_{pt} \cdot r_p + \sum_{q=p+1}^{I}\left(y_{pqt} \cdot r_{pq} + \sum_{w=q+1}^{I} y_{pqwt} \cdot r_{pqw}\right)\right) \cdot c_r \tag{3}$$

Subject to:

$$S_{ij}^0 + \sum_{t=1}^{z}\sum_{k=1}^{K} x_{ijtk} \cdot d_k - \sum_{t=1}^{z-1} q_{ij} \le Q_{ij} \qquad \forall \mathrm{i} \in \mathrm{I} \;\; \forall \mathrm{j} \in \mathrm{J} \;\; \forall \mathrm{z} \in \mathrm{T} \tag{4}$$

$$S_{ij}^0 + \sum_{t=1}^{z}\sum_{k=1}^{K} x_{ijtk} \cdot d_k - \sum_{t=1}^{z} q_{ij} \ge q_{ij} \qquad \forall \mathrm{i} \in \mathrm{I} \;\; \forall \mathrm{j} \in \mathrm{J} \;\; \forall \mathrm{z} \in \mathrm{T} \tag{5}$$

$$y_{pt} \le H_{pt} \qquad \forall \mathrm{t} \in \mathrm{T} \;\; \forall \mathrm{p} \in \mathrm{I} \tag{6.1}$$

$$2 \cdot y_{pqt} \le H_{pt} + H_{qt} \qquad \forall \mathrm{t} \in \mathrm{T} \;\; \forall (\mathrm{p,q}) \in \mathrm{I}^2 \text{ for all } \mathrm{p} < \mathrm{q} \tag{6.2}$$

$$3 \cdot y_{pqwt} \le H_{pt} + H_{qt} + H_{wt} \qquad \forall \mathrm{t} \in \mathrm{T} \;\; \forall (\mathrm{p,q,w}) \in \mathrm{I}^3 \text{ for all } \mathrm{p} < \mathrm{q} < \mathrm{w} \tag{6.3}$$

$$H_{it} \le \sum_{j=1}^{J}\sum_{k=1}^{K} x_{ijtk} \qquad \forall \mathrm{t} \in \mathrm{T} \;\; \forall \mathrm{i} \in \mathrm{I} \tag{7}$$

$$H_{it} \ge \frac{1}{J \cdot K} \cdot \sum_{j=1}^{J}\sum_{k=1}^{K} x_{ijtk} \qquad \forall \mathrm{t} \in \mathrm{T} \;\; \forall \mathrm{i} \in \mathrm{I} \tag{8}$$

$$\sum_{p=1}^{I}\left(y_{pt} + \sum_{q=p+1}^{I}\left(2 \cdot y_{pqt} + \sum_{w=q+1}^{I} 3 \cdot y_{pqwt}\right)\right) = \sum_{i=1}^{I} H_{it} \qquad \forall \mathrm{t} \in \mathrm{T} \tag{9}$$

$$y_{it} + \sum_{q=i+1}^{I}\left(y_{iqt} + \sum_{w=q+1}^{I} y_{iqwt}\right) + \sum_{p=1}^{I}\sum_{i=p+1}^{I}\left(y_{pit} + \sum_{w=i+1}^{I} y_{piwt}\right) + \sum_{p=1}^{I}\sum_{q=p+1}^{I}\sum_{i=q+1}^{I} y_{pqit} \le 1 \qquad \forall \mathrm{t} \in \mathrm{T} \quad \forall \mathrm{i} \in \mathrm{I} \tag{10}$$

$$\sum_{j=1}^{J}\sum_{k=1}^{K} x_{ijtk} \cdot k \le K \qquad \forall \mathrm{t} \in \mathrm{T} \;\; \forall \mathrm{i} \in \mathrm{I} \tag{11}$$

$$\sum_{j=1}^{J}\sum_{k=1}^{K}\left(x_{pjtk} + x_{qjtk}\right) \cdot k \le K \cdot \left(2 - y_{pqt}\right) \qquad \forall \mathrm{t} \in \mathrm{T} \;\; \forall (\mathrm{p,q}) \in \mathrm{I}^2 \text{ for all } \mathrm{p} < \mathrm{q} \tag{12}$$

$$\sum_{j=1}^{J}\sum_{k=1}^{K}\left(x_{pjtk} + x_{qjtk} + x_{wjtk}\right) \cdot k \le K \cdot \left(3 - 2 \cdot y_{pqwt}\right) \qquad \forall \mathrm{t} \in \mathrm{T} \quad \forall (\mathrm{p,q,e}) \in \mathrm{I}^3 \text{ for all } \mathrm{p} < \mathrm{q} < \mathrm{w} \tag{13}$$

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T}\sum_{k=1}^{K} x_{ijtk} \cdot k = K \cdot \left(\sum_{t=1}^{T}\sum_{p=1}^{I}\left(y_{pt} + \sum_{q=p+1}^{I}\left(y_{pqt} + \sum_{w=q+11}^{I} y_{pqwt}\right)\right)\right) \tag{14}$$

$$H_{it}, x_{ijtk}, y_{pqwt}, y_{pqt}, y_{pt} \in \{0,1\} \quad \forall \mathrm{j} \in \mathrm{J} \; \forall \mathrm{t} \in \mathrm{T} \; \forall \mathrm{k} \in \mathrm{K} \; \forall \mathrm{i} \in \mathrm{I} \; \forall (\mathrm{p,q,w}) \in \mathrm{I}^3 \text{ for all } \mathrm{p} < \mathrm{q} < \mathrm{w} \tag{15}$$

Constraints (4) limit the maximum quantity of fuel up to the reservoir capacity in each of the observed time periods, and constraints (5) define the minimum quantity of fuel in reservoirs that can meet the demand in the observed time period. Constraints (6.1) to (6.3) define that a route can be performed if and only if all of the petrol stations included in the route are to be supplied on that day. Inequalities (7) and (8) define petrol stations that must be served. A petrol station must be served if at least one compartment of at least one fuel type should be delivered. Constraints (9) assure that the number of stations served by all of the routes is equal to the number of all of the stations that need to be served. Constraints (10) limits the number of routes visiting petrol station $\boldsymbol{i}$ in time period $\boldsymbol{t}$ to only one. Constraints (11) prohibit multiple direct deliveries (vehicles serving only one station in a single route) to the same petrol station during the same day. If constraints (11) are omitted, then different fuel types can be delivered to the same station, each in a quantity lower than four compartments. However, when the total number of compartments for all of the fuel types delivered to the same station is greater than four, then the station is to be visited more than once in the same time period. Constraints (12) and (13) restrict the delivery quantity in each route that serves more than one station to a maximum of four compartments in each day of the planning horizon. Constraint (14) assures that all vehicles are fully loaded. Constraints (15) define the binary nature of the variables.

4 HEURISTIC MODELS

The IRP consists of two sub-problems that are often in conflict. The routing problem is one of those two, which implies that it is impossible to obtain the solution from MILP models even for moderate size problems. Some of them developed the approach based on the delivery frequencies that are dependent on fuel consumption in a station, and the vehicle capacity. Also, in the case when total volume demanded by each station in a given time period is close to the vehicle capacity, direct shipping can perform very well; on the contrary, the multi-stop deliveries can have much better performances. We observe two heuristic approaches: the direct delivery and the multi-stop delivery based on the frequencies of inter-shipment. For the purpose of the route construction we used two basic methods that were also analyzed by Derigs et al. (2010) and proven to be effective for vehicle routing with compartments: the Savings method, and the Sweep method. Also, we developed three intra-period search procedures for routes improvement, and the inter-period search procedure for solution improvement by reallocation of all deliveries for a certain station, one or more days earlier in a planning horizon; we use the VND to guide the search procedures. In the following text we will describe two heuristic approaches and the VND guided search.

In the case of the high intensity of demands from single stations, the direct delivery approach can be very effective. The main idea is to determine, starting from the first day of the planning horizon, which stations will have inventory level of a fuel type below its consumption, and to insert those stations in the delivery plan for that day. Because all vehicles must be fully loaded, we add a compartment to a fuel type with the smallest ratio of required inventory level and realized consumption. After each added compartment we update the inventory level for that fuel type, and repeat the procedure until the vehicle is fully loaded.

When single station have small utilization ratio of vehicle capacity, multi-stop delivery has better performance. We have allowed routes with up to three stops, where each stop represents a single station. Since the fuel consumption is assumed to be deterministic, inter-shipment frequencies for each station may be easily determined. Petrol stations have more than one fuel type and therefore have more than one possible frequency. This was the reason to develop two heuristic sub-models based on inter-shipment frequencies: "*per station frequency*" heuristic where the inter-shipment frequency of a petrol station equals the smallest fuel type frequency from that station; "*per fuel type frequency*" heuristic where we observe the inter-shipment frequency of every fuel type. In the latter case, time instants of delivery are determined for each fuel type, individually in each petrol station. The inter-shipment frequency $\boldsymbol{f_{ij}}$ for fuel type $\boldsymbol{j}$ in petrol station $\boldsymbol{i}$ is calculated by Eq. (16), where the dividend represents the maximum vehicle capacity that can be allocated to a single fuel type. The inter-shipment frequency $\boldsymbol{F_i}$ for petrol station $\boldsymbol{i}$ is calculated by Eq. (17).

$$f_{ij} = \left\lfloor \left\lfloor \frac{K}{J} \right\rfloor \cdot d_1 \Big/ q_{ij} \right\rfloor \quad \forall i \in I \quad \forall j \in J \tag{16}$$

$$F_i = \min_{j \in J} \{ f_{ij} \} \quad \forall i \in I \tag{17}$$

The VND guided search, in general, means that the first occurrence of improvement in neighborhood search is accepted and the search is repeated until no more improvements can be found. For the local intra-period search, we used three neighborhood structures to improve the routing of deliveries on each day of the planning horizon: (1) the interchange of a single station between two routes for the same day; (2) the removal of a single station from one route and its insertion into another route for the same day; and (3) 2-opt*, the removal of two arcs (one from each route) and finding the best possible reconnection. For the inter-period search we used the approach of removing all of the delivery quantities for a single station from one day and insertion into another day where that station already has some deliveries.

5 COMPUTATIONAL RESULTS

We have generated 50 small scale problem instances (10 petrol stations and 4 day planning horizon) to compare the MILP and heuristic models, and 50 moderate scale problem instances (50 petrol stations and 7 day planning horizon) to evaluate the heuristic models since the MILP model is not capable to solving the moderate scale problems in reasonable computational time. We observe two cases considering vehicle type: small capacity vehicles (3 compartments of 6t capacity); high capacity vehicles (5 compartments of 6t capacity). Each petrol station is supplied with $\boldsymbol{J}$=3 different fuel types. Stock levels of fuels $\boldsymbol{S^0_{ij}}$ at the beginning of observed interval are generated randomly from the interval [6, 12]. The daily fuels consumptions on stations $\boldsymbol{q_{ij}}$ are assumed to be constant over the planning horizon and a consumption values are chosen randomly (40% of stations consume from the interval [1, 2] t/day, 40% of stations consume from the interval [2, 3] t/day, and 20% of stations consume from the interval [3, 6] t/day). The reservoirs capacities $\boldsymbol{Q_{ij}}$ have 40t capacity. The spatial coordinates of petrol stations are randomly generated in square 50 by 50 km while the depot is located in the center of that square. The daily cost of carrying inventory is $\boldsymbol{c_{inv}}$=1 €/t and the cost for the one traveled kilometer is $\boldsymbol{c_r}$=2 €/km. The MILP models were implemented through the CPLEX 12 on a desktop PC with a 2.0 GHz Dual Core processor with 2 GB of RAM memory. The heuristics were implemented in Python 2.6.

In addition to the full triple assignment MILP model (MILP-3), we have considered the cases when each route can have maximum two stops and when only direct deliveries are allowable. The comparison results of MILP and heuristic models for small scale problem instances are presented in the Table 1. The results obtained from the heuristic models for 50 moderate scale problem instances are presented in the Table 2.

Table 1 The comparison of the MILP and heuristic models; 50 small scale problem instances

	3x6t vehicle					5x6t vehicle					Avg. calc. time (sec.)
	Total IRP costs Avg.	Dist. to the best average solution	Total IRP costs Stdev.	Avg. Routing costs	Avg. inventory costs	Total IRP costs Avg.	Dist. to the best average solution	Total IRP costs Stdev.	Avg. Routing costs	Avg. inventory costs	
MILP-3	2929.6	0.00%	524.5	1973.0	956.6	2385.7	0.00%	364.2	1354.9	1030.8	644.9
MILP-2	2939.9	0.35%	518.0	1980.4	959.5	2425.3	1.66%	357.1	1379.5	1045.8	46.9
MILP-1	3224.0	10.05%	577.8	2203.7	1020.3	2888.6	21.08%	423.9	1670.1	1218.6	0.5
Direct delivery heuristic	3240.7	10.62%	579.4	2215.0	1025.7	2898.6	21.50%	423.4	1675.2	1223.4	1.66
"*per station frequency*" Sweep Big Gap heuristic	3061.6	4.51%	571.8	2083.8	977.8	2645.3	10.88%	406.9	1551.0	1094.4	1.73
"*per station frequency*" Savings heuristic	3060.5	4.47%	569.3	2083.8	976.8	2638.6	10.60%	401.7	1547.8	1090.8	1.69

"*per fuel type frequency*" Sweep Big Gap heuristic	3067.7	4.72%	570.2	2098.6	969.2	2611.8	9.48%	384.8	1535.4	1076.4	1.74
"*per fuel type frequency*" Savings heuristic	3057.7	4.38%	570.3	2084.9	972.8	2624.4	10.01%	400.7	1543.3	1081.2	1.74

Table 2 The results obtained from the heuristic models; 50 moderate scale problem instances

	3x6t vehicle					5x6t vehicle					Avg. calc. time (sec.)
	Total IRP costs Avg.	Dist. to the best average solution	Total IRP costs Stdev.	Avg. Routing costs	Avg. inventory costs	Total IRP costs Avg.	Dist. to the best average solution	Total IRP costs Stdev.	Avg. Routing costs	Avg. inventory costs	
Direct delivery heuristic	30216	5.92%	2198	21105	9111	25890	15.60%	1535	14712	11178	2.3
"*per station frequency*" Sweep Big Gap heuristic	28526	0.00%	2266	20202	8324	23010	2.74%	1464	14087	8923	246
"*per station frequency*" Savings heuristic	28543	0.06%	2268	20339	8204	22396	0.00%	1538	13939	8457	954
"*per fuel type frequency*" Sweep Big Gap heuristic	29286	2.67%	2274	20383	8903	22966	2.55%	1581	13378	9588	26
"*per fuel type frequency*" Savings heuristic	29302	2.72%	2259	20400	8902	22874	2.14%	1533	13422	9452	51

6 CONCLUSIONS

In this paper we have evaluated the quality of the direct delivery heuristic and the heuristic with multi-stop delivery based on the frequencies of inter-shipment, for solving the IRP. The multi-stop heuristics produces better results than the direct delivery heuristic for both vehicle type cases, and both small and moderate scale problems, as it is presented in the Tables 1 and 2. For the small scale problem instances, multi-stop "*per fuel type frequency*" heuristics with the Savings method produces the best average results for 3x6t vehicle, and multi-stop "*per fuel type frequency*" heuristics with the Sweep Big Gap method produces the best average results for 5x6t vehicle. For the moderate scale problem instances, multi-stop "*per station frequency*" heuristics with the Savings method produces the best average results for 3x6t vehicle, and multi-stop "*per station frequency*" heuristics with the Sweep Big Gap method produces the best average results for 5x6t vehicle. One of the possible future research directions should be concentrated on the impact of different distribution structural parameters to the IRP models performances (such as spatial density, depot location, consumption intensity, etc.).

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MICROECONOMIC MODEL INSTRUMENTS FOR THE ANALYSIS OF A COMPETITIVE ENVIRONMENT STATE IN SLOVAKIA

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Abstract
The paper focuses on the presentation of microeconomic model instruments for the analysis of a competitive environment state in Slovakia. In connection with the transformation of the Slovak economy to a market model was necessary to deal within a relatively short period of time with series of urgent and important tasks that were connected with new principles of the market forces operation. A natural assumption of the proportionate economic development is a protection of the economic competition. Conditions of the economic competition in the world developed countries and the European Union countries as well are maintained and controlled institutionally, and this systemic element of economic development is of a great importance. In Slovakia the realization of tasks related to the protection of the economic competition is carried out by the Antimonopoly Office.
The paper describes ways of utilizing model approaches, economic and mathematical methods and computing technology when evaluating the current state and the development of the competitive environment in Slovakia.

Keywords: *Antimonopoly office, competitive environment, relative and absolute concentration, concentration ratio, Herfindahl index, Coefficient of variation*

JEL Classification: C44

1 INTRODUCTION

Tasks related to the protection of competition in the Slovak Republic are performed by Antimonopoly Office of the Slovak Republic which has, in its relatively short period of existence, focused on preparation of and later enforcement of rigorous adherence to the Act No. 188/94 Coll. on Protection of the Economic Competition [7].
One of the most important attributes of flawless, and above all harmonically developing, economic system is protection of the principles of competition guaranteed by the state. Violation of these principles, which is of course, from the point of view of the actors themselves, profitable and may be, in some cases, even effective for the society in a short run, is, considering the long run aspect of development of economy, undoubtedly negative trend.
This article presents basic concepts of methodology of sectoral enterprise concentration measurements. We explain the methods of quantification of degree of enterprise concentration, indicators of absolute and relative concentration while characterizing the extreme cases of sectoral concentration.
Antimonopoly office of the Slovak Republic, as well as related institutions in the European Union countries, aim to carry out methodical analysis of the state of competitive environment and, with in the developed industrial countries commonly applied exact methods, to quantify degree of concentration in individual sectors of national economy and on the basis of this analysis to prepare qualified information for the executive institutions for following two levels of decision making:

- development of economic policy conception from the long term perspective and
- its current correction from the short term perspective [4].

2 METHODS FOR CALCULATION OF SECTORAL CONCENTRATION DEGREE

There are many methods published in the literature which can be more or less successfully applied for evaluation of the degree and impact of concentration in the conditions of imperfect competition (e.g. [1], [2], [6]). Essence of most of these methods lies in quantification of indicators which, in a way, describe the position of individual manufacturer within sector on relevant market of a given commodity or characterize state of competitive environment in given sector.
Selection of particular type of relevant market concentration characteristic or indicator is determined by the aim of particular analysis in the particular conditions of the sector and particular group of goods. Depending on whether the indicators quantify the degree of concentration with regard to all subjects or only to their subset with certain characteristic, we can divide the indicators into two groups [3], namely:

a) **Indicators for measurement of absolute concentration**, among which belong for example:

- a1) *concentration ratio* defined for specific number (e.g. 3, 6, 10, 25 and 50) of subjects in sector with highest value of watched indicator,
- a2) *Herfindahl index*; which systematically characterizes degree of concentration in a sector
- a3) *dominance index*, which is used concurrently with Herfindahl index in cases when market subjects merge on the relevant market

b) **Indicators for measurement of relative concentration**, among which belong for example:

- b1) *dispersion rate*, which refers to concentration ratio congruent with individual groups of subjects with the highest value of watched indicator,
- b2) *coefficient of variation*, which represents degree of distribution of influence of all subjects in terms of watched indicator,

Basic indicator of degree of concentration is ratio of individual company (or firm) production on production of the sector or production of specifically defined set of companies. Its analytical form if following:

$$r_k = \frac{q_k}{Q} = \frac{q_k}{\sum_{j=1}^{n} q_j} \qquad (1)$$

where

n - number of companies in the sector,
q_k – quantity of production of a company k, k=1,...,n,
Q - quantity of production of the sector,
r_k – ratio of company k on production of the sector.

In cases of so called equal distribution of watched indicator among all manufacturing subjects in the industry, i.e. in case following is valid

$$q_k = q_{k+1} = \frac{Q}{n} k = 1,\dots,n-1$$

the market share of individual manufacturers is constant and following holds

$$r_k = \frac{q_k}{Q} = \frac{\frac{Q}{n}}{Q} = \frac{1}{n} \qquad k = 1,\dots,n$$

2.1 Concentration ratio

Concentration ratio CR_ψ is the indicator of degree of so called *absolute* concentration in a sector and it is calculated for ψ “biggest” companies in terms of market share of homogenous production, in other words it represents ratio of first ψ companies with the highest quantity of production on production of all companies in a sector

$$CR_{\psi} = \frac{1}{Q} \times \sum_{k=1}^{\psi} q_k \tag{2}$$

whereas $\psi \in \langle 1, n \rangle$ and for the quantities of production q_k of companies following inequation holds

$$q_p > q_{p+1}, \quad \text{for } p=1, 2, \ldots, n-1$$

(it means that companies are, for the purposes of the analysis, arranged decreasingly in relation to the watched indicator) and coefficients of concentration ratio take value from interval

$$0 \leq CR_{\psi} \leq 1$$

Coefficient of ratio or concentration ratio is usually quantified for the market share of companies turnover in relation to the overall production of the sector. However there are analogical analyses possible also for other economic characteristics.

2.2 Herfindahl index

Herfindahl or Herfindahl-Hirschman index for evaluation of absolute concentration in a sector currently belongs among the standardly used methodology for analysis of degree of sectoral concentration for example in the USA as well as in Germany. Index takes into account number of companied in the sector as well as their market share. Construction of Herfindahl index is based on the hypothesis that importance of company in a sector is function of square of its market share.
This philosophy of Herfindahl index construction obviously highlights the influence of "strong" subjects and vice versa eliminates the influence of "small" manufacturers. Analytical representation of Herfindahl index of concentration is following:

$$H = h(q_1, q_2, \ldots, q_n) = \sum_{k=1}^{n} \left(\frac{q_k}{Q} \right)^2 = \sum_{k=1}^{n} r_k^2 \tag{3}$$

where

h – real function of n-variables, h: $R^n \rightarrow R$,
- meaning of other identifiers is analogical as in (1).

Interesting are also the extreme values of Herfindahl index. Lower border of Herfindahl index is not constant; it is the function of number of companies in a sector. Assuming *the market share of each of the companies on overall turnover of sector is equal* Herfindahl index includes minimal value H_d and thus following holds

$$q_k = \frac{Q}{n} \qquad k = 1, 2, \ldots, n \tag{4}$$

After substitution of assumption (4) into the relation (3) we get

$$H_d = h(q_1, q_2, \ldots, q_n) = \sum_{k=1}^{n} \left(\frac{q_k}{Q} \right)^2 = \sum_{k=1}^{n} \left(\frac{\frac{Q}{n}}{Q} \right)^2 = n \times \frac{1}{n^2} = \frac{1}{n} \tag{5}$$

On the other hand upper border of Herfindahl index is $H^h = 1$, whereas this extreme case occurs only if we assume *the existence of pure* monopoly in sector, i.e. $n=1$, $q=Q$ and after substituting into the relation (3) we get

$$H^h = h(q) = \sum_{k=1}^{1} \left(\frac{q_k}{Q} \right)^2 = \left(\frac{Q}{Q} \right)^2 = 1$$

For the needs of practical analyses the Herfindahl index is not used in a form (3), as a number from the interval $\langle 1/n; 1 \rangle$ (in sectors with high number of companies the value of Herfindahl index in definition expression according to relation (3) would be very low), but after *multiplication with appropriate multiplier*, which corresponds to the percentage expression of market shares of individual companies. For example the *Monopolies commission of the Federal Republic of Germany* as well as *Federal Statistical Office of Germany* use multiplication factor 1000.

Usual classification of degree of sectoral concentration according to the value of Herfindahl index is following [3]:

- *Not concentrated industry* – value of Herfindahl index $H < 1000$,
- Moderately concentrated industry – value of Herfindahl index falls into interval $H \in \langle 1000, 1800)$,
- *Concentrated industry* – value of Herfindahl index $H \geq 1800$.

2.3 Dominance Index

Dominance index complements Herfindahl index and it is effectively used in concurrence with it. Dominance index is utilized mainly in cases when several subjects on relevant market merge. In this case Herfindahl index increases but the dominance index does not necessarily increase as well. Its analytical form is following:

$$DI = d(q_1, q_2, ..., q_n) = \sum_{k=1}^{n} \frac{r_k}{H}^2 = \sum_{k=1}^{n} \frac{r_k^2}{\sum_{k=1}^{n} r_k^2} = \frac{1}{H} \sum_{k=1}^{n} \left(\frac{q_k}{Q} \right)^2 = \frac{1}{H} \sum_{k=1}^{n} \left(\frac{q_k}{\sum_{k=1}^{n} q_k} \right)^2 \tag{6}$$

Like the Herfindahl index the value of dominance index is in the case of monopoly on the relevant market equals to 1 and in case of equal shares of subject on the market equals to 1/n.

2.4 Dispersion Rate

Dispersion rate DR_ψ represents ratio of how the value of concentration ratio ψ of subjects with the highest value of watched indicator influences the inequality of distribution of values of watched indicator of individual subjects. Analytical form of this indicator is following:

$$DR_\psi = \frac{CR_\psi - \frac{\psi}{n}}{CR_\psi} = 1 - \frac{\psi}{n \times CR_\psi} \tag{7}$$

where meaning of used symbols is analogical to the definitions of relations (1), (2), (3).
In cases of equal distribution of market shares of individual subjects in industry, for the dispersion rate holds:

$$DR_\psi = \frac{CR_\psi - \frac{\psi}{n}}{CR_\psi} = 1 - \frac{\psi}{n \times CR_\psi} = 1 - \frac{\psi}{n \times \frac{\psi}{n}} = 1 - 1 = 0$$

After specification of dispersion rate DR_ψ we can of course calculate the concentration ratio ψ of biggest subjects on overall number n of subjects of aggregate

$$CR_\psi = \frac{\psi}{n \times (1 - DR_\psi)} \tag{8}$$

When the dispersion rate DR_ψ as an indicator for the measurement of relative concentration ratio represents certain equivalent to the indicator of absolute concentration ratio CR_ψ, then the equivalent of relative concentration ratio for Herfindahl index is *coefficient of variation*.

2.5 Coefficient of Variation

Coefficient of variation represents very effective tool for evaluation of degree of concentration while taking into consideration, which is substantive, exact number of subjects of watched aggregate.
In comparison with Herfindahl index, which reacts to the square of individual concentration ratios r_k, and in consequence does not reflect the influence of less important subjects of aggregate, the coefficient of variation represents ratio of distribution of influence of all subjects from the point of view of watched indicator, whereas the average value or arithmetic average of individual

influence ratio of individual subjects is considered the reverse value of their overall number, i.e. figure *1/n*. Analytical form of the coefficient of variation is following:

$$V^2 = n \times \sum_{j=1}^{n} \left(r_j - \frac{1}{n} \right)^2 \tag{9}$$

On the basis of relation (9) the value of Herfindahl index can be expressed as the function of coefficient of variation and overall number *n* of subjects of aggregate as follows

$$H = \frac{\left(1 + V^2\right)}{n} \tag{10}$$

In the case of absolutely equal distribution of market share of individual subjects in the sector the value of coefficient of variation is equal to zero, because

$$r_j = \frac{1}{n} \qquad \forall j = 1, \ldots, n$$

$$V^2 = n \times \sum_{j=1}^{n} \left(r_j - \frac{1}{n} \right)^2 = n \times \sum_{j=1}^{n} \left(\frac{1}{n} - \frac{1}{n} \right)^2 = 0$$

3 CONCLUSION

Basic indexes presented here as well as others measures published in the literature enable the qualified assessment of degree of sectoral concentration. There are common conventions, e.g. American or German, which set the border values of these indexes, i.e. values, if they are exceeded, indicating the eventuality of restriction of competition.

Equally important field of use of this methodology is, besides the already mentioned analysis of current state of sectoral concentration, evaluation of perspective impacts of sectoral concentration of subjects under review. Then this, or analogical method, could be used as an effective supplementary criterion for decision making on approval of concentration.

However we have to realize that process to achieve high propositional ability, analytical capability, complexity and information value of studies on the development and current state of competitive environment of individual industries in the Slovak Republic will take a long time and require many years of systematic and conceptual work as well as solid organizational, material as well as personal resources. Systematic and theoretically well done work in this field is equally necessary and useful for execution of rational and efficient competition policy in the Slovak Republic.

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MODELS FOR COMBINED OVERBOOKING AND CAPACITY CONTROL IN NETWORK REVENUE MANAGEMENT

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Abstract
Revenue management is the art and science of enhancing firm revenue while selling the same quantity of product. Network revenue management models attempt to maximize revenue when customers buy bundles of multiple resources. Overbooking and capacity control and are two fundamental pieces of revenue management. Revenue management practices often include overbooking capacity to account for customers who make reservations but later make cancellations or do not show up. Capacity control deals with the question of which requests should be accepted over time. Overbooking and capacity control decisions interact. The joint problem can be formulated as a dynamic program. This dynamic program cannot be solved by using traditional dynamic programming tools for problems of practical size. The paper analyzes linear approximations of the joint problem of overbooking and capacity control in network revenue management.

Keywords: *network revenue management, overbooking, capacity control, approximations*

JEL Classification: C44
AMS Classification: 90B50

1 INTRODUCTION

Revenue Management (RM) is the art and science of predicting real-time customer demand and optimizing the price and availability of products according to the demand (see Talluri, van Ryzin, 2004, Philips, 2005). The RM area encompasses all work related to operational pricing and demand management. This includes traditional problems in the field, such as dynamic pricing, capacity allocation, overbooking and others. Recent years have seen great successes of revenue management, notably in the airline, hotel, and car rental business. Currently, an increasing number of industries is exploring to adopt similar concepts. What is new about RM is not the demand-management decisions themselves but rather how these decisions are made. The true innovation of RM lies in the method of decision making.
Overbooking was the first RM tool that appeared in the operations research literature (Taylor, 1962) and the first to be implemented in practice. Overbooking is a response to the fact that customers who order product for future delivery often fail to show up to collect at the time when the product becomes available. Often the “no-show” is the customer’s decision, but “no-shows” may also be the result of operational factors. Network revenue management models attempt to maximize revenue when customers buy bundles of multiple resources. The dependence among the resources in such cases is created by customer demand. The basic model of the network revenue management problem is formulated as a stochastic dynamic programming problem whose exact solution is computationally intractable. There are several approximation methods for the problem. The Deterministic Linear Programming (DLP) method is a popular in practice. The DLP method is based on an assumption that demand is deterministic and static. The paper analyzes linear approximations of the joint problem of overbooking and capacity control in network revenue management. The model combines the DLP model with a single period overbooking model.

2 OVERBOOKING

Overbooking is concerned with increasing capacity utilization in a reservation-based system when there are significant cancelations or no-shows (see Phillips, 2005). A reservation is a forward contract between a customer and the firm. Firms are adopting business practices designed to reduce no-show rates.

Overbooking is applicable in industries with the following characteristics:

- Booking are accepted for future use.
- Capacity is constrained.
- Customers are allowed to cancel or not show.
- The cost of denying service is relatively low.

The standard practice to respond to no-shows is simply to overbook or to accept more orders for future delivery of a product that there is product available. The potential revenue gain is especially significant if the product is perishable and cannot be held in inventory. The basic overbooking problem is then to decide how many orders to accept for future delivery of a product based on the number of units that will be available. An intelligent decision requires historical data and estimates of no-show rates.

There are some overbooking policies based on different objective functions. A simple deterministic heuristic that calculates an overbooking limit based on capacity and expected no-show rate. A service-level policy involves managing to specific target, for example maximal rate of admissible denied services. A cost-based policy involves explicitly estimating costs of denied service and comparing those costs with potential revenue to determine the overbooking levels that maximize expected total revenue minus expected overbooking costs.

A hybrid policy is one in which cot-based limits are calculated but constrained by service-level restrictions. A cost-based policy is proposed for the combined model.

The cost-based approach requires an estimate of the revenue loss not accepting additional reservations and an estimate of the cost of denied service. Suppose z customers show up on the day of service, and let $d(z)$ denote the denied-service cost function, assuming that there is an increasing convex function of z. A common assumption in practice is that each denied-service costs a constant marginal amount h. Let c denote the physical capacity, then

$$d(z) = h(z - c)^+.$$

Let r denote the marginal revenue generated by accepting an additional reservation. A common simplification in practice is to consider it fixed. Let y denote the number of reservations on hand, and the random variable $z(y)$ denotes the number of customers who show up on the day of service out of y reservations. The total expected net revenue is given by

$$V(y) = ry - E[d(z(y))].$$

The simplest model is based on a binomial model of cancellations in which no-shows are lumped together with cancelations. Let denote q the probability that a reservation currently on hand show up at the time of service. Under some assumptions (see Talluri, van Ryzin, 2004), the show demand is binomially distributed with the probability mass function

$$p(z(y) = z) = \binom{y}{z} q^z (1-q)^{y-z}, \ z = 0, 1, \dots, y,$$

and the distribution function

$$F_y(z) = p(z(y) \le z) = \sum_{k=0}^{z} \binom{y}{k} q^k (1-q)^{y-k},$$

with mean $E[z(y)] = qy$ and variance $\text{var}(z(y)) = yq(1-q)$.

The optimal booking limit x^* is the largest value of x satisfying the condition

$$\Delta V(y) = E[d(z(x))] - E[d(z(x-1))] \le r.$$

Fort the binomial model, this condition reduces to $hqp(z(x-1) \ge c) \le r$,
and it can be rewritten to the condition

$$1 - F_{x-1}(c-1) \le \frac{r}{qh}.$$

3 NETWORK CAPACITY CONTROL

The quantity-based revenue management of multiple resources is referred as network revenue management. This class of problems arises for example in airline, hotel, and railway management. In the airline case, the problem is managing capacities of a set of connecting flights across a network, so called a hub-and-spoke network. In the hotel case, the problem is managing room capacity on consecutive days when customers stay multiple nights.

Network revenue management models attempt to maximize revenue function when customers buy bundles of multiple resources. The interdependence of resources, commonly referred to as network effects, creates difficulty in solving the problem.

The basic model of the network revenue management problem can be formulated as follows (see Talluri, van Ryzin, 2004):

The network has m resources which can be used to provide n products. We define the incidence matrix $\mathbf{A} = [a_{ij}]$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, where

$a_{ij} = 1$, if resource i is used by product j, and

$a_{ij} = 0$, otherwise.

The j-th column of $\mathbf{A}$, denoted $\mathbf{a}_j$, is the incidence vector for product j. The notation $i \in \mathbf{a}_j$ indicates that resource i is used by product j.

The state of the network is described by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ of resource capacities. If product j is sold, the state of the network changes to $\mathbf{x} - \mathbf{a}_j$.

Time is discrete, there are T periods and the index t represents the current time, $t = 1, 2, \ldots, T$. Assuming within each time period t at most one request for a product can arrive.

Demand in time period t is modeled as the realization of a single random vector $\mathbf{r}(t) = (r_1(t), r_2(t), \ldots, r_n(t))$. If $r_j(t) = r_j > 0$, this indicates a request for product j occurred and that its associated revenue is r_j. If $r_j(t) = 0$, this indicates no request for product j occurred. A realization $\mathbf{r}(t) = \mathbf{0}$ (all components equal to zero) indicates that no request from any product occurred at time t. The assumption that at most one arrival occurs in each time period means that at most one component of $\mathbf{r}(t)$ can be positive. The sequence $\mathbf{r}(t)$, $t = 1, 2, \ldots, T$, is assumed to be independent with known joint distributions in each time period t. When revenues associated with product j are fixed, we will denote these by r_j and the revenue vector $\mathbf{r} = (r_1, r_2, \ldots, r_n)$.

Given the current time t, the current remaining capacity x and the current request $\mathbf{r}(t)$, the decision is to accept or not to accept the current request. We define the decision vector $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))$ where

$u_j(t) = 1$, if a request for product j in time period t is accepted, and

$u_j(t) = 0$, otherwise.

The components of the decision vector $u(t)$ are functions of the remaining capacity components of vector x and the components of the revenue vector r, $u(t) = u(t, x, r)$. The decision vector $u(t)$ is restricted to the set

$$U(x) = \{u \in \{0, 1\}^n,\ \mathrm{A}u \le x\}.$$

The maximum expected revenue, given remaining capacity x in time period t, is denoted by $V_t(x)$. Then $V_t(x)$ must satisfy the Bellman equation

$$V_t(\mathbf{x}) = E\left[\max_{u \in U(x)} \{\mathbf{r}(t)^T u(t, \mathbf{x}, \mathbf{r}) + V_{t+1}(\mathbf{x} - \mathbf{Au})\}\right] \tag{1}$$

with the boundary condition $V_{T+1}(\mathbf{x}) = 0, \forall \mathbf{x}$.

A decision $\mathbf{u}^*$ is optimal if and only if it satisfies:

$u_j(t, \mathbf{x}, r_j) = 1$, if $r_j \ge V_{t+1}(\mathbf{x}) - V_{t+1}(\mathbf{x} - \mathbf{a}_j)$, $\mathbf{a}_j \le \mathbf{x}$,

$u_j(t, \mathbf{x}, r_j) = 0$, otherwise.

This reflects the intuitive notion that revenue r_j for product j is accepted only when it exceeds the opportunity cost of the reduction in resource capacities required to satisfy the request.

The classical technique of approaching this problem has been to use a deterministic LP solution to derive policies for the network capacity problem. Initial success with this method has triggered considerable research in possible reformulations and extensions, and this method has become widely used in many industrial applications.

The Deterministic Linear Programming (DLP) method uses the approximation

$$V_t^{LP}(\mathbf{x}) = \max \mathbf{r}^T \mathbf{y}$$

Subject to
$$\mathbf{Ay} \leq \mathbf{x} \tag{2}$$
$$\mathbf{0} \leq \mathbf{y} \leq E[\mathbf{D}]$$

where $\mathbf{D} = (D_1, D_2, \ldots, D_n)$ is the vector of demand over the periods t, $t+1, \ldots, T$, for product j, $j = 1, 2, \ldots, n$, and $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ is the vector of revenues associated with the n products. The decision vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ represent partitioned allocation of capacity for each of the n products. The approximation effectively treats demand as if it were deterministic and equal to its mean $E[\mathbf{D}]$. The optimal dual variables, $\boldsymbol{\pi}^{\mathrm{LP}}$, associated with the constraints $\mathbf{Ay} \leq \mathbf{x}$, are used as bid prices.

The DLP was among the first models analyzed for network RM. The main advantage of the DLP model is that it is computationally very efficient to solve. Due to its simplicity and speed, it is a popular in practice. The weakness of the DLP approximation is that it considers only the mean demand and ignores all other distributional information. The performance of the DLP method depends on the type of network, the order in which fare products arrive and the frequency of re-optimization.

4 COMBINED MODELS

Models for coordinating network-capacity controls and overbooking decisions are analyzed. The main question is how to set overbooking levels on a network. The capacities of network resources are key inputs to capacity-control problems. Using overbooking, these capacities may be inflated defining virtual capacities for each resource that exceed the physical capacity. This increase in capacity affects the accept or reject decisions of the capacity-control method. On the other hand, capacity-control decisions clearly influence the opportunity cost of capacity, which is a key input to economic overbooking models. Hence, the total revenue for a network is affected both by overbooking and capacity-control practices. Despite the strong interdependence of these decisions, the two problems are typically separated in practice.

The time horizon will be divided into two periods: a reservation period, and a service period. The reservation period spans (0, T] and the reservations can be made for any of the n products. The reservation period is followed by the service period, during which the customers with reservations show up or become no-shows. During the service period, the firm may deny service to customers who show up in case of insufficient capacity, in which case it pays a penalty.

One way to formulate this overbooking problem is as a two-stage, static model that combines the DLP model and the cost-based overbooking models The demand of reservation requests arrive according to a stochastic process during (O, T]. The problem parameters are $E[\mathbf{D}]$, the vector of expected demand to come for the n classes. Let $\mathbf{c} = (c_1, ..., c_m)$ denote the vector of resource capacities, and the vector $\mathbf{r} = (r_1, ..., r_n)$ denote the vector of revenues associated with the n products. There is a denied-service cost on each resource given by the vector $\mathbf{h} = (h_1, ..., h_m)$. The denied-service cost may differ from one resource to another, but it does not vary with time of product type. Decision variables are $\mathbf{x}$, the vector of overbooking levels (virtual capacities) and $\mathbf{y}$, the vector of primal capacity allocations. The show demand for resource i in this formulation is approximated by the random variable $z_i(x_i)$, vector $\mathbf{z}(\mathbf{x})$ is the vector of show demand.

The formulation is as follows:

$$V(\mathbf{x},\mathbf{y}) = \max(\mathbf{r}^T\mathbf{y} - E\left[\mathbf{h}^T(\mathbf{z}(\mathbf{x}) - \mathbf{c})^+\right])$$

Subject to
$$\mathbf{Ay} \leq \mathbf{x} \tag{3}$$
$$\mathbf{0} \leq \mathbf{y} \leq E[\mathbf{D}]$$
$$\mathbf{x} \geq \mathbf{c}$$

The objective function is the total net revenue, revenue minus denied-service costs. Let $R(\mathbf{y}) = \mathbf{p}^{\mathrm{T}}\mathbf{y}$ denote the revenue function and $C(\mathbf{x}) = E[\mathbf{h}^{\mathrm{T}} (\mathbf{z}(\mathbf{x}) - \mathbf{c})^+]$ denote the overbooking-cost function. The overbooking-cost function $C(\mathbf{x})$ is a non-decreasing and convex function of the overbooking limit **x** if the random variable associated with the of survivors for resource i, $z_i(x_i)$ is assumed to follow the binomial model with survival probability q_i. Thus, the objective function of the problem (3) is jointly concave in **y** and **x.**

A general nonlinear programming method can be used to solve the problem. An alternating-direction method (see Bertsekas and Tsitsiklis, 1997) algorithm specialized to this problem's structure.

5 CONCLUSION

The paper analyzes linear approximations of the joint problem of overbooking and capacity control in network revenue management. The model combines the deterministic linear programming (DLP) model with a single period overbooking model. The DLP method was used for simplicity. The used approximation great1y simplifies the model and is a good approximation in the important case where demand is high. The approach can be adapted to other network approximations as well (such as RLP and PNLP) (see Fiala, 2010). The same formulation applies to a variety of network bid-price methods. A cost-based policy of overbooking with binomial model of cancellation was proposed for the combined model.

Poisson model of cancellation or other policies of overbooking are possible to use.

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DOES FOREIGN DIRECT INVESTMENT AFFECT ECONOMIC GROWTH? EVIDENCE FROM OECD COUNTRIES.

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Abstract
The literature concerning the foreign direct investment (FDI) and economic growth generally points to a positive FDI effects on economic growth. However, the theory provides conflicting predictions concerning the economic growth – FDI relation. In theory, economic growth may induce FDI inflow, and FDI may also stimulate economic growth. The aim of this paper is to add to the empirical literature, we test the relationship between FDI and the rate of growth of GDP using a stochastic frontier model and employing panel data covering OECD countries over the period 9 years.

Keywords: *Foreign Direct Investment, Economic growth, Stochastic Frontier Analysis*

JEL Classification: C23, E22
AMS Classification: 91B82, 91B84

1 INTRODUCTION

The literature on FDI and economic growth generally points to a positive relationship between the two variables. In general, economists agree that foreign direct investment inflows lead to an increased rate of economic growth. FDI may affect economic growth, through its impact on capital stock, technology transfer, skill acquisition, or market competition and in addition as foreign direct investment is often accompanies by advanced technology domestic investors can also adopt this advanced technology. Particularly FDI should exert positive effects on economic growth in developing countries which suffer from low productivity and capital stock deficiencies. On the other hand, FDI and growth may also exhibit a negative relationship, particularly if the inflow of FDI leads to increased monopolization of local industries, thus compromising efficiency and growth dynamics. In theory we can also find reverse relationship, economic growth may induce FDI inflow, when FDI is seeking consumer markets. Many empirical studies attempt to identify causality between FDI and GDP and the estimation of the relationship between these variables has been an issue amongst empirical economists for some time. This increased interest issues from mixed empirical evidence. Some authors have argued that GDP growth induces FDI while other observers believe this relationship is reversed. For example, Borensztein et al. [7] detected positive impact FDI on GDP, but the magnitude of this effect depends on the level of human capital. De Mello [11] also supposed positive effect FDI on GDP, but the contributions of FDI depends on primarily host country characteristics, mainly the quantum of skilled labor. Johnson [17] demonstrated that FDI inflows increased economic growth in developing countries, but not in advanced nations. The effects of FDI on the economic growth have been shown to be both positive in numerous other empirical studies ([13], [14], [25]) and negative relationship ([21], [28]). Generally, the positive growth effects of FDI have been more likely when FDI is drawn into competitive markets, whereas negative effects on growth have been more likely when FDI is drawn into heavily protected industries. The opposite link between economic growth and FDI, the positive effect of host country economic growth on FDI inflow has been also confirmed by various studies ([27], [2], [15], [24], [25]).

This study attempts to make a contribution in this context to the empirical literature on the relationships between economic growth and FDI inflows in host countries. As most of the empirical studies in this field have employed the least squares method to examine the relationship between examined variables, our paper exploits a stochastic production function approach. Based on the endogenous growth model where the output is a function of the standard factors of production plus human capital we estimated stochastic frontier model. Our model contains input factor variables as well as several variables as inefficiency variables. This approach allows us to distinguish the effects of FDI on economic growth via technical and efficiency change and quantify the effects of FDI, along with other variables, on efficiency levels. The exploitation of this approach in this context is not very frequent and can be found e.g. in the studies of Iyer et al. [16] or Wijeweera [29].

2 METHODOLOGY FOR ANALYZING FDI AND ECONOMIC GROWTH RELATIONSHIP

Stochastic frontier analysis has become a popular tool for production analysis. Stochastic frontier models date back to studies of Aigner, Lovell and Schmidt [1] and Meesen and van den Broek [20], who independently proposed a stochastic frontier production function with a two-part composed error term. In the production context (this approach could be also used in cost context), where its use is most common this error is composed of a standard random error term, representing measurement error and other random factors, and a one-sided random variable representing technical inefficiency, i.e. the distance of the observation from the production frontier. This technical efficiency reflects the ability of a unit (firm, country or schools) to obtain maximal output from a given set of inputs. If the unit is 100 % efficient, it lay on the production frontier itself and this measure is bounded between zero and one.

In our study a stochastic frontier production function is applied to panel data to examine whether FDI inflows enhances economic growth via efficiency gains and to estimate technical efficiency for the selected countries. Our analysis follows SFA model of Battese and Coelly [6] called Technical Inefficiency Effects Model. The general form of the stochastic frontier production function model for panel data can be formulated as follows:

$$Y_{it} = f(X_{it}, \beta)\exp(v_{it} - u_{it}) \quad (1)$$

where Y_{it} denotes the production at the *t*-th observation $(t = 1,2,...T)$ for the *i*-th unit $(i = 1,2,...N)$, X_{it} is the corresponding matrix of explanatory variables, β is a vector of unknown parameters to be estimated, v_{it} is symmetric random variable, u_{it} is time variant technical inefficiency term of compound error term $\varepsilon_{it} = v_{it} - u_{it}$. In this specification the error term is composed of two uncorrelated parts. The first part u_{it} is capturing the effect of technical inefficiency and the second part v_{it} is reflecting effect of statistical noise. Usually we assume that v_{it} are random variables to be normally distributed ($v_{it} \sim$ iid N(0, σ_v^2)) and u_{it} are non-negative time-invariant random variables to be half normal distributed ($u_{it} \sim$ iid $N^+(0, \sigma_u^2)$) or truncated normal distribution ($u_{it} \sim$ iid $N^+(\mu,\sigma_u^2)$) can be also considered. The technical inefficiency effects u_{it} in the stochastic frontier model (1) could be also specified by equation (2), where u_{it} are assumed to be a function of a set of explanatory variables:

$$u_{it} = \mathbf{z}_{it}^T\delta + w_{it} \quad (2)$$

where $\mathbf{z}_{it}$ is a vector of explanatory variables associated with technical inefficiency of production of unites over time, u_{it} are non-negative random variables, associated with technical inefficiency of production, which are defined by normal distribution: $u_{it} \sim$ iid $N^+(\mathbf{z}_{it}^T\delta, \sigma_u^2)$, δ is a vector of unknown parameters to be estimated and the random variable w_{it} is defined by the truncation of the normal distribution: $w_{it} \sim$ N(0, σ_w^2), such that the point of the truncation is $-\mathbf{z}_{it}^T\delta$. The parameters of both the stochastic frontier model and the inefficiency effects model can be consistently estimated by using maximum likelihood estimation (MLE) method. The next step is to obtain estimates of the technical efficiency of each unit. The problem is to extract the information that ε_{it} contains on u_{it} (we have estimates of $\varepsilon_{it} = v_{it} - u_{it}$, which obviously contain information on u_{it}). A solution to the problem is obtained from the conditional distribution of u_{it}

given ε_{it}, which contains whatever information ε_{it} contains concerning u_{it}. This procedure is known as JLMS decomposition (for more details see [18]). For separation the inefficiency effect from the statistical noise can be also used an alternative minimum squared error predictor estimator (for more details see [19]). Once the point estimates of u_{it} are obtained, estimates of the technical efficiency of each unit can be obtained by substituting them into equation (3). Given the specification in (1) and (2), the appropriate measure of individual technical efficiency of production for *i*-th unit in the *t*-th year is defined by:

$$TE_{it} = \exp\{-u_{it}\} \tag{3}$$

3 MODEL SPECIFICATION AND DATA

Model defined by equations (1) and (2) was applied to find whether FDI enhances efficiency in the host economy. Our unbalanced panel data set of 30[1] OECD countries observed over a period from 2002 to 2010 includes 266 observations in total. This data panel includes many of the world's most advanced countries but also emerging countries like Mexico or Turkey. The choice of this sample was driven by our attempt to include an economically diverse set of countries. All data are based on information from statistics of OECD and the Transparency International. One of the main steps of the analysis is to choose appropriate form of the production function. We decided to use more flexible translog function instead of more traditional Cobb – Douglas production function due to the fact that Cobb – Douglas specification imposes severe restrictions of the technology by restricting the production elasticities to be constant and the elasticities of the input substitution to be unity.

As we indicated in the first part of the paper some studies argued that relationship between GDP and FDI follows a two-way causality. Once FDI is used as the dependent variable, on the other hand other works treats with GDP as the dependent variable. In order to make a decision concerning GDP and FDI relationship we conducted a Granger Causality Test for FDI and GDP. The test results have confirmed a strong one-way causality from real GDP to FDI, consequently the suitability for GDP choice as the dependent variable. Therefore, the dependent variable of our model represents the real GDP (*Y*) in term of millions of USD. Our model includes two types of variables: factor inputs and inefficiency variables. Standard factors of production such as capital (*C*) and labor (*L*) are included in the factor input section. Capital is expressed as Gross Capital Formation in constant 2000 USD millions and labor is defines as civilian labor force in thousands. The input section of our model also includes interaction terms of explanatory variables, a linear and non-linear time trends. As FDI (in USD millions) inflows may increase efficiency in a host economy via several ways we have chosen this variable as the main inefficiency variable. In addition some more variables are included among inefficiency variables. The Corruption Perceptions Index – CPI (*Corruption*) was chosen to control for institutional inefficiency, index varies from 0 (highly corrupt) to 10 (highly clean). The country's trade (*Openness*) is expressed by the sum of exports and imports in USD millions. Thus, our model can be written as follows:

$$\begin{aligned}\ln Y_{it} &= \beta_0 + \beta_1 \ln C_{it} + \beta_2 \ln L_{it} + (1/2)\left[\beta_{11}(\ln C_{it})^2 + \beta_{22}(\ln L_{it})^2 + \beta_{12}(\ln C_{it})(\ln L_{it})\right] + \\ &+ \beta_{13}(\ln C_{it})t + \beta_{23}(\ln L_{it})t + \beta_{33}t^2 + v_{it} - u_{it}\end{aligned} \tag{4}$$

where

$$u_{it} = \delta_1 \ln FDI_{it} + \delta_2 \ln Corruption_{it} + \delta_3 \ln Openness_{it} + w_{it} \quad i = 1,\dots,N \quad t = 1,\dots,T \tag{5}$$

[1] Nowadays OECD consists of 34 members (Australia, Austria, Belgium Canada, Chile, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Israel, Italy, Japan, Korea, Luxembourg, Mexico, Netherlands, New Zealand, Norway, Poland, Portugal, Slovak Republic, Slovenia, Spain, Sweden, Switzerland, Turkey, United Kingdom, United States); our data set contains 30 countries which were members during whole observed period 2002 – 2010 (we excluded Chile, Estonia, Israel, Slovenia).

The parameters of the model defined by equation (4) and (5) have been jointly estimated by the maximum likelihood estimation method using FRONTIER 4.1 [9]. For the separation the inefficiency effect from the statistical noise was used Battese and Coelli point estimator (see [9]). Individual technical efficiency estimates were obtained by substituting the inefficiency effects to the equation (3). The final estimates of the parameters are listed in Table 1. Applied model also provides individual efficiency measures for the countries in each year but it is not possible to present these extensive results due to insufficient space.

Table 1 Parameters of the Production Function and Inefficiency Effects; Source: own calculations (Frontier 4.1)

	Coefficients	Standard Error	t-Ratios
Frontier Model			
Constant	3,8120*	1,2470	3,0569
Capital	0,4848*	0,0954	5,0824
Labor	-0,2400*	0,1001	-2,3976
Capital^2	-0,0210*	0,0043	-4,8495
Labour^2	0,0977*	0,0101	9,6619
Capital x Labor	0,0089*	0,0051	1,7533
Capital x Trend	0,0017*	0,0010	1,8007
Labor x Trend	0,0001	0,0022	0,0290
Trend^2	-0,0019*	0,0009	-2,1424
Inefficiency Effects			
FDI Inflows	0,3709*	0,0251	14,7627
Corruption	-0,5291*	0,0634	-8,3440
Openness	-0,3755*	0,0311	-12,0893
Variance Parameters			
Sigma-squared	0,0397*	0,0066	5,9847
Gamma	0,9593*	0,0209	45,9208
Log-likelihood	174,5476		
LR-Test	384,7087		

* significant at $\alpha = 0,05$

4 CONCLUSION

A majority of the parameters concerning production factors are statistically significant at conventional levels and have expected positive signs besides negative signs of the parameters of Labor and quadratic term for Capital. We have included quadratic trend term to capture the time trend, it is statistically significant but it does not have expected positive sign.

The gamma parameter value (for more details see [10]) is 0,96 which suggests that 96 % of the disturbance term is due to inefficiency. The results of performed LR test (for more details see [10]) confirms the presence of technical inefficiency, therefore we conclude that the technical inefficiency term is a significant addition to the model. Inefficiency term in our model is expressed as explicit vector function of specific explanatory variables associated with technical inefficiency called inefficiency effects variables. Negative signs of these parameters mean an increase in efficiency and a positive effect on economic growth. However, our results do not confirm our expectations as for the impact of FDI inflows on economic growth. In our model FDI parameter is statistically significant but has positive sign (i.e. negative impact on growth). We have assumed that FDI inflows should enhance economic growth. One possible explanation for this opposite result confirmed by our model has been proposed by Hanson [29], he argued that multinational firms could confine domestic firms to less profitable ventures creating productivity losses. If these losses are greater than the corresponding productivity gains created by multinational investment, then we can expect an aggregate negative impact on economic growth and in addition FDI inflows can crowds out domestic investments. Our results suggest that FDI inflows have a negative effect on the economic growth but this conclusion may not be so straightforward e.g. it should be interesting to observe interaction between FDI and high quality educated labor (due to unavailability of data we could not to include this variable to the model). Study of Wijeweera et.al. [28] confirmed that FDI inflows exert a positive impact on economic growth only in presence of a highly skilled labor i.e. FDI by itself does not induce

efficiency gains. This is an important finding, especially for developing countries with unskilled labor. Another inefficiency variable Openness has negative sign and thus has positive effect on economic growth. Various studies confirmed this result and show that countries that have chosen to open their economies over the last two decades have achieved considerably higher growth compared to countries that remained comparatively closed [29]. Our last finding concerns corruption; the model indicates that less corruption in the host economy would increase economic growth. The corruption can depress economic activity, decrease of FDI inflows or inhibit impact of FDI.
The technical efficiency indexes (calculated by equation (3)) reflect how far a country is from its best possible production frontier. TE statistics over a period 2002 – 2010 suggest that countries such United States, United Kingdom, Canada, Australia or France are closest to the production frontier (their TE scores are higher than 0,90).

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DISCRETE HAMILTONIAN PROBLEMS WITH IP-SOLVER

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Abstract
The cheapest Hamiltonian Circle or the cheapest Hamiltonian path problems belong to the family of NP-hard problems and usage of a commercial IP-solver to solve their common instances seems hopeless. Nevertheless, we present two special types of the above-mentioned problems, where additional constraints and thorough model building enable to use a commercial IP-solver to obtain an optimal solution of some problem instances from transportation practice.

Keywords: *Hamiltonian Circle, NP-hard problem, public transport, coordination of bus arrivals, free order of objects.*

JEL Classification: C61
AMS Classification: 90C27

1 INTRODUCTION

A classical case of the discrete Hamiltonian problem is the task to find the cheapest circle in a weighted complete graph so that the circle contains all the graph vertices. This problem is known as the traveling salesman problem and many researchers tackled it several decades ago [3, 8]. They developed many special solving techniques and achieved great successes [1] in solving large instances of the problem. In comparison to general solving techniques embedded to various commercial optimization environments, the mentioned special solving techniques are able to solve instances, which are in several order larger than those ones solvable by general solving techniques. Nevertheless, a general IP-solver can be useful, when it is necessary to solve some case of discrete Hamiltonian problem with additional conditions [7]. We present here the traveling salesman problem with time windows and use the special features of the associated model to complete a model of the bus link coordination problem with free order of arrivals at a given bus stop. We shall present some results of numerical experiments to demonstrate abilities of a general optimization environment in this area of applications.

2 THE TRAVELLING SALESMAN PROBLEM WITH TIME WINDOWS

In this problem a starting node of an agent (travelling salesman) is given and the agent should visit all remaining nodes 2, …, m in an arbitrary order and return back to the starting node. Traversing an edge connecting nodes i and k costs d_{ik} and takes a positive time t_{ik}. Each node k=2, …,m has its individual time window $< a_k, b_k >$, in which it must be visited by the agent. The agent should complete its trip so that the total cost of traversed edges is minimal. It is obvious that the order of node visits determines both feasibility and the total cost of the trip.
To model the order of visits, we introduce zero-one variables w_{ik}, where the variable takes the value of one if and only if the agent visits node k directly after visiting i. The variables w_{ik} are introduced only for so-called relevant pairs, i.e. for such pairs of nodes, where node i can directly precede node k. To make the following model more concise, we define a logical function *exists*(i, k), which takes the value of true exactly if the variable w_{ik} is defined. For example, the value of *exists*(i, i) is always equal to false. Furthermore we introduce a variable t_k for each node k = 2, …, m, which denotes the time of agent's visit at the node. After these preliminaries, a model of the travelling salesman problem with time windows can be formulated as follows.

$$\textit{Minimize} \quad \sum_{i=1}^{m} \sum_{\substack{k=1 \\ exists(i,k)}}^{m} d_{ik} \cdot w_{ik} \tag{1}$$

$$\textit{Subject to} \quad \sum_{\substack{i=1 \\ exists\,(i,\,k)}}^{m} w_{ik} = 1 \qquad for\ k = 1,...,m \tag{2}$$

$$\sum_{\substack{i=1 \\ exists\,(i,\,k)}}^{m} w_{ki} = 1 \qquad for\ k = 1,...,m \tag{3}$$

$$t_k - t_i \geq t_{ik} - T(1 - w_{ik}) \quad for\ i = 2,...,m;\ k = 2,...,m;\ exists\,(i,\,k) \tag{4}$$

$$t_k \geq a_k \qquad for\ k = 2,..,m \tag{5}$$

$$t_k \leq b_k \qquad for\ k = 2,..,m \tag{6}$$

$$w_{ik} \in \{0, 1\} \qquad for\ i = 1,..,m,\ k = 1,...,m;\ exists\,(i,\,k) \tag{7}$$

In the model above, the consistency constraints (2) and (3) ensure that each node *k* has exactly one predecessor and successor in an agent's trip. The constraints (4), where *T* is enough large value, assure that if a time of visit at *i* directly precedes the time of visit at *k*, then the time gap between the two visits must be greater or equal to t_{ij}. These constraints together with supposed positive values of t_{ij} serve also as the sub-tour breaking constraints, which prevent any feasible solution from being formed by more than one circle. Constraints (5) and (6) allow the time of visits only in the associated time windows. It is obvious that the structure of the time window set influences the complexity of the solved problem.

The complexity can vary from *NP*-complexity of the original traveling salesman problem to simple identification of a path in the associated network. As large instances the first case are hardly solvable by a general optimization environment, the instances of the other case can be easily solvable by any solving technique to optimality. This possibility evoked us an idea to make use of a general optimization environment for solving the following case of Hamiltonian discrete problem.

3 BUS LINK COORDINATION PROBLEM WITH FREE ORDER OF ARRIVALS

In the problem, *n* arrivals of vehicles at a stop in the designate period are considered. Let t_i be arrival time of vehicle *i* at the stop. The earliest possible arrival time of the vehicle *i* is denoted as a_i. This time may be postponed until the time $b_i = a_i + c_i$ is reached, where c_i is the maximum possible shift of arrival at the stop. It is necessary to find such time positions of the individual arrivals so that the total passengers' waiting time is minimal.

The first and last arrival time t_0 and t_n are fixed. The aim is to shift times t_i for $i = 1, \dots, n-1$, so that the total waiting time of passengers in passenger-minutes is minimal. The total waiting time of passengers in the period $< t_0, t_n >$ can be expressed as:

$$\sum_{i=1}^{n} \frac{1}{2} f\,(t_k - t_{i(k)})^2 \tag{8}$$

It is assumed that passengers come at the stop uniformly with intensity *f* and the index *i(k)* denotes index of the arrival time, which directly precedes the arrival time t_k. Previous approaches [6] were focused on the problem of time coordination with fixed order of bus link arrivals. Here, we deal with the problem of time coordination with free order of bus arrivals. In this case a rearrangement of order of bus arrivals at a given bus stop is possible. To model the free order of arrivals, we follow approach from the previous section and introduce zero-one variables w_{ik} for $i = 0, \dots, n-1$, $k = 1, \dots, n$, which model whether link *i* directly precedes link *k* or not. These variables will be defined only for pairs (i, k) of links, where the direct preceding is possible, what is expressed by the logical function *exist* defined in the previous section. Then mathematical model of the problem with free order of arrivals of bus links will be as follows:

$$\textit{Minimize} \quad \frac{1}{2} f \sum_{k=1}^{n} \sum_{j=1}^{m(k)} (2j-1)\cdot u_{kj} \tag{9}$$

$$\textit{Subject to} \quad x_k + a_k - t_0 \le \sum_{j=1}^{m(k)} u_{kj} + T\cdot(1-w_{0k}) \quad \textit{for } k = 1,..,n-1 \textit{; exists } (0, k) \tag{10}$$

$$x_k + a_k - t_0 \ge \sum_{j=1}^{m(k)} u_{kj} - T\cdot(1-w_{0k}) \quad \textit{for } k = 1,..,n-1 \textit{; exists } (0, k) \tag{11}$$

$$x_k + a_k - x_i - a_i \le \sum_{j=1}^{m(k)} u_{kj} + T\cdot(1-w_{ik}) \quad \textit{for } i = 1,..,n-1;\ k = 1,..,n-1; \textit{ exists } (i,k) \tag{12}$$

$$x_k + a_k - x_i - a_i \ge \sum_{j=1}^{m(k)} u_{kj} - T\cdot(1-w_{ik}) \quad \textit{for } i = 1,..,n-1;\ k = 1,..,n-1; \textit{ exists } (i,k) \tag{13}$$

$$t_n - x_i - a_i \le \sum_{j=1}^{m(n)} u_{nj} + T\cdot(1-w_{in}) \quad \textit{for } i = 1,..,n-1 \textit{; exists } (i, n) \tag{14}$$

$$t_n - x_i - a_i \ge \sum_{j=1}^{m(n)} u_{nj} - T\cdot(1-w_{in}) \quad \textit{for } i = 1,..,n-1 \textit{; exists } (i, n) \tag{15}$$

$$\sum_{\substack{i=0 \\ exists(i,\,k)}}^{n-1} w_{ik} = 1 \quad \textit{for } k = 1,..,n \tag{16}$$

$$\sum_{\substack{k=1 \\ exists(i,\,k)}}^{n} w_{ik} = 1 \quad \textit{for } i = 0,..,n-1 \tag{17}$$

$$x_i \ge 0 \quad \textit{for } i = 1,..,n-1 \tag{18}$$

$$x_i \le c_i \quad \textit{for } i = 1,..,n-1 \tag{19}$$

$$u_{ij} \ge 0 \quad \textit{for } i = 1,..,n,\ j = 1,..,m(i) \tag{20}$$

$$u_{ij} \le 1 \quad \textit{for } i = 1,..,n,\ j = 1,..,m(i) \tag{21}$$

$$w_{ik} \in \{0,1\} \quad \textit{for } i = 0,..,n-1,\ k = 1,..,n \textit{; exists } (i, k) \tag{22}$$

We approximate the quadratic function (8) by a piecewise linear function (9) as it is mentioned in [2, 4, 5]. In this case the constant $m(k)$ for $k = 1, \ldots, n$ denotes the number of dividing points of the approximation of the quadratic function (1). We consider that each of the time intervals is divided by minutes. Then we can compute these constants as:
$m(k) = b_k - t_0$ for $k = 1, \ldots, n-1$ and $m(n) = t_n - t_0 = T$. The non-negative auxiliary variables u_{kj} for $k=1, \ldots, n, j=1, \ldots, m(k)$ are used as substitution (23) for $t_k - t_{i(k)}$.

$$t_k - t_{i(k)} = \sum_{j=1}^{m(k)} u_{kj} \quad \textit{for } k=1,..,n \tag{23}$$

Under constraints (21), a variable u_{kj} represents covering of the j-th unit interval by the value of t_k- $t_{i(k)}$. The square of t_k can then be approximated by the expression (24).

$$(t_k - t_{i(k)})^2 = \sum_{j=1}^{m(k)} (2j-1)\cdot u_{kj} \quad \textit{for } k=1,..,n \tag{24}$$

If the values of u_{kj} are integer, then the right-hand-side of (24) represents the exact value of $(t_k)^2$. Similarly to the model (1) – (7) presented in the previous section, the consistency constraints (16) and (17) ensure that each arrival k has exactly one predecessor and successor excepting the fixed arrivals. The constraints (10) – (15), where T is enough large value, assure that the substitution

(23) is valid, i.e. the gaps $x_k+a_k-x_i-a_i$ correspond exactly to the differences between arrival t_k and the directly preceding arrival $t_{i(k)}$.

4 NUMERICAL EXPERIMENTS

To demonstrate usefulness and solvability of the above-formulated Hamiltonian discrete problems for transport practice, we have chosen real problems of the public transport in the selected area of the Czech Republic. Each of the solved instants corresponds to the problem (9) – (21) with free order arrivals of nine bus links to an observed bus stop. As the matter of question of these NP-hard problems is the computational time necessary for optimal solution, we focus on mapping the relation between the computational time *CT* and the number of introduced variables w_{ik}. Three sequences of the numerical experiments are reported. The first sequence called ”Basic” uses full set of the 72 variables w_{ik}, where $i \neq k$ and the associated results are plotted into the first pair of rows in table 1. The second sequence of experiments denoted as “Reduced” was performed with such relevant variables w_{ik}, where the associated pair *i, k* of arrivals satisfies the constraint $a_i < a_k + c_k$. The third sequence – “Advanced” uses only such variables w_{ik}, where the arrivals *i, k* satisfy above constraint and, in addition, there were no arrival *j*, for which the following inequalities hold:
All results are reported in the table 1 in the corresponding rows, where the computational time necessary for obtaining an exact solution of the problem is given in seconds.

Table 1 Computational times and the number of introduced variables w_{ij}

	Name	1	2	3	4	5	6	7
Basic	w_{ij}	72	72	72	72	72	72	72
	CT	0.641	0.407	0.406	0.468	1.422	0.891	0.734
Reduced	w_{ij}	49	48	47	47	56	53	50
	CT	0.609	0.344	0.344	0.422	1.094	0.781	0.578
Advanced	w_{ij}	23	20	20	19	38	33	28
	CT	0.235	0.187	0.297	0.281	1.015	0.688	0.266

The experiments were performed using the optimization software Xpress-IVE. The associated code was run on a PC equipped with the Intel Core 2 6700 processor with parameters: 2.66 GHz and 3 GB RAM.

5 CONCLUSIONS

Even if the size of used benchmarks is not too large, the solved instances were obtained from real public transport situations and thus we can state that the suggested approach can be useful for improving distribution of bus arrivals at a stop in some interval and reduce waiting time of passengers, which come at the stop randomly. Comparison of the reported sequences of experiments shows that even if the underlying problem is complex in general, there is a possibility to solve bigger instances if the input data are properly analyzed and the number of allocation variables is reduced. During our experiments we have noticed that if he constant *T* is set at its maximal relevant value, then the lower bounds at start of the branch-and-bound method is too low. This constitutes another way of improving efficiency of the presented approach, when bigger instants are solved.

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EXPECTATIONS OF BAIL OUT AND COLLECTIVE MORAL HAZARD

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Abstract
We analyze infinite horizon stochastic game with discounting of future single period payoffs that models interaction between Cournot duopolists producing differentiated products. There are two states of demand for output of each firm. The probabilities of their occurrence depend on random disturbances affecting demand and investing or not investing into product innovation. If the low state of demand occurs for output of each firm the government bails out both of them. Taking into account expected bailout, there exists a strong perfect equilibrium in which, along the equilibrium path, the firms collude on not investing into product innovation. In a strong perfect equilibrium, no coalition in no subgame can strictly Pareto improve the vector of average discounted payoffs of its members.

Keywords: *bail out, innovation, moral hazard, stochastic game, strong perfect equilibrium.*

JEL Classification: C73, D43.
AMS Classification: 91A06.

1 INTRODUCTION

The current economic crisis prompted governments to various forms of support of firms (e.g., in the form of scrapping subsidy aimed at support of car makers). Although in some cases (like the scrapping subsidy – see [5] for the analysis of its impact) such a support was an indirect one, we will use the term “bail out” for it. (During the crisis, even the European Commission is willing to approve government aid that it would ban in more prosperous times.) Bail outs can be expected especially when all or most of the firm in an industry are in trouble. In such a case, it can be argued that the firms themselves are not responsible for their bad results. Nevertheless, expectations of bail out can reduce the endeavors of firms to help themselves, especially through product or technological innovation. In particular, if none of the firms in the industry invests in research and development (henceforth, R&D), expected average discounted profit of each of them can be (thanks to bail outs) higher than if all of them invested in R&D. In such a case, it is in the interest of all firms to cooperate in suppressing R&D. If one of them deviates, invests in R&D and its expenditures lead to product (or technological) innovation, the other firms can do the same, but in the way that punishes the deviator (e.g., by a significant increase in the output of innovated product). That is, a deviation does not lead to competition in innovation, but to innovation by the other firms aimed at punishing the deviator. In the present paper, we illustrate this behavior by a simplified game theoretic model.

2 MODEL

We denote the analyzed game by Γ . Its time horizon is N .There are two players in it. They are Cournot duopolists that produce differentiated products. The goods produced by them are substitutes. For firm $j \in \{1,2\}$, the set of possible states of demand for its product is $\Omega_j = \{\omega_{j1}, \omega_{j2}\}$. We set $\Omega = \Omega_1 \times \Omega_2$. For each $j \in \{1,2\}$, the inverse demand function

$P_j : [0,\infty)^2 \times \Omega_j \to [0, P_j(0,0,\omega_{j2})]$ assigns to each vector of outputs of duopolists $y = (y_1, y_2) \in [0,\infty)^2$ and each state of the demand $\omega_j \in \Omega_j$ the unit price $P_j(y,\omega_j)$ at which the demand for j's product equals y_j.

Assumption 1: For each $j \in \{1,2\}$, (i) P_j is continuous on its domain, (ii) P_j is nonincreasing in y_1 as well as in y_2 on its domain, (iii) P_j is strictly decreasing in y_1 as well as in y_2 at each $(y,\omega_j) \in [0,\infty)^2 \times \Omega_j$ with $P_j(y,\omega_j) > 0$, (iv) P_j is concave in y at each $(y,\omega_j) \in [0,\infty)^2 \times \Omega_j$ with $P_j(y,\omega_j) > 0$, (v) P_j is twice continuously differentiable with respect to y at each $(y,\omega_j) \in (0,\infty)^2 \times \Omega_j$ with $P_j(y,\omega_j) > 0$, (v) the first order partial derivatives with respect to y do not depend on ω_j, (vi) $P_j(y,\omega_{j1}) < P_j(y,\omega_{j2})$ for each $y \in [0,\infty)^2$, and (vii) for each $\omega_j \in \Omega_j$, there exists $y_j^{\max}(\omega_j)$ such that $P_j(y,\omega_j) = 0$ for each $y \in [0,\infty)^2$ with $y_j \geq y_j^{\max}(\omega_j)$ and $y_j^{\max}(\omega_{j2}) > y_j^{\max}(\omega_{j1})$.

We assume (without loss of generality of the model) that each firm $j \in \{1,2\}$, in each period, at each $\omega_j \in \Omega_j$ chooses its output from the interval $[0, y_j^{\max}(\omega_j)]$.

Each firm $j \in \{1,2\}$, in each period $t \in N$, decides whether to invest into R&D aimed at product innovation. In order to keep the model as simple as possible, we distinguish only two decisions on R&D investment: 0 (not investing into R&D) and 1(investing into R&D). The cost of investing into R&D for firm $j \in \{1,2\}$ is $b_j > 0$. The outcome of R&D investment in each $j \in \{1,2\}$ is stochastic. If it successful, it shifts the graph of the inverse demand function for j's product upwards, i.e., it leads to a higher state of the demand. We denote state of the demand for product of firm $j \in \{1,2\}$ in period $t \in N$ by $\omega_j(t)$; $\omega(1) = (\omega_1(1), \omega_2(1))$ is given. For each $j \in \{1,2\}$, function $\mu_j : \Omega_j^2 \times \{0,1\} \to [0,1]$ assigns to each $(\omega_j(t), \omega_j(t+1), I_j(t)) \in \Omega_j^2 \times \{0,1\}$ the probability of occurrence of state of the demand $\omega_j(t+1)$ in period $t+1$ when state of the demand in period t is $\omega_t(t)$ and investment into R&D in period t is $I_j(t)$. The probability distributions specified by μ_1 and μ_2 are independent.

Assumption 2: For each $j \in \{1,2\}$, (i) $1 > \mu_j(\omega_{j1}, \omega_{j2}, 1) > \mu_j(\omega_{j1}, \omega_{j2}, 0) = 0$,
(ii) $1 > \mu_j(\omega_{j2}, \omega_{j2}, 1) > \mu_j(\omega_{j2}, \omega_{j2}, 0) = 0$, (iii) $\mu_j(\omega_{j2}, \omega_{j2}, 1) > \mu_j(\omega_{j1}, \omega_{j2}, 1)$.

Assumption 3: Cost function of each firm $j \in \{1,2\}$, $c_j : [0,\infty) \to [0,\infty)$, is (i) twice continuously differentiable, (ii) strictly increasing, and (iii) convex.

Both firms observe the state of demand for both of them in each period before making decisions on output and R&D investment. Each firm observes ex post the competitors output (i.e., each $j \in \{1,2\}$, at the beginning of each period $t \in N \setminus \{1\}$, knows the output of firm $i \neq j$ in all preceding periods). The government also observes the state of demand for each firm at the beginning of each period. Neither the competitor nor the government observes a firm's investment into R&D. In each period $t \in N$, in which $\omega_j(t) = \omega_{j1}$ for each $j \in \{1,2\}$, the government pays to each firm subsidy $M > 0$. If the latter condition is not satisfied, the government pays no subsidy to the firms.

Both players discount their future single period payoffs by the common discount factor $\delta \in (0,1)$. We denote the set of non-terminal (i.e., finite) histories in Γ by H and the set of terminal (i.e., infinite) histories by Z. $H = \bigcup_{t \in N} H(t)$, where $H(t)$ is the set of histories leading to period $t \in N$. $H(1)$ contains only the initial (i.e., empty) history denoted by $\varnothing$. For each $t \in N \setminus \{1\}$, each

$h \in H(t)$ has the form $h = \{(\omega_1(\tau), \omega_2(\tau), y_1(\tau), y_2(\tau), I_1(\tau), I_2(\tau))\}_{\tau=1}^{t-1}$, where, for each $j \in J$ and each $\tau \in \{1,...,t-1\}$, $\omega_j(\tau)$ is state of the demand for j's output in period τ, $y_j(\tau)$ is firm j's output in period τ, and $I_j(\tau)$ is its investment into R&D in period τ. We denote the set of non-terminal public histories by H_p. We have $H_p = \bigcup_{t \in N} H_p(t)$, where $H_p(t)$ is the set of public histories leading to period t. For $t \in N \setminus \{1\}$, the elements of $H_p(t)$ differ from the elements of $H(t)$ only by not including firms' investments into R&D. $H_p(1)$ includes only the empty public history.

We restrict attention to pure Markov strategies in Γ. (See [3] and [4] for characterization of Markov strategies.) A pure Markov strategy of firm $j \in \{1,2\}$ is a function that assigns to each $(h, \omega_j) \in H_p \times \Omega_j$ a pair $(y_j(h, \omega_j), I_j(h, \omega_j)) \in [0, y_j^{\max}(\omega_j)] \times \{0,1\}$. The firms make decisions on output and investment into R&D simultaneously. When a firm makes a decision on its output and its investment into R&D in period $t \in N$, it knows the public history $h \in H_p(t)$, as well as states of the demand for both firms in period t. We denote the set of pure Markov strategies of firm $j \in \{1,2\}$ in Γ by S_j and set $S = S_1 \times S_2$. For each $j \in \{1,2\}$, $\pi_j : S \to \Re$ is firm j's payoff function in Γ defined on the set of profiles of pure Markov strategies. It assigns to each $s \in S$ j's average discounted profit (that includes also government subsidies) generated by it. (In order to keep the notations as simple as possible, we do not explicitly indicate the dependence of the average discounted profit on the discount factor in the symbol for firm's payoff function in Γ.) That is, if terminal history $h = \{(\omega_1(\tau), \omega_2(\tau), y_1(\tau), y_2(\tau), I_1(\tau), I_2(\tau))\}_{\tau=1}^{\infty}$ is the outcome of $s \in S$, then

$$\pi_j(s) = (1-\delta)\left[\sum_{\tau \in N} \delta^{\tau-1}\left[P_j(y(\tau), \omega_j(\tau))y_j(\tau) - c_j(y_j(\tau)) - b_j I_j(\tau)\right] + M \sum_{\tau \in N : \omega_1(\tau) = \omega_{11} \& \omega_2(\tau) = \omega_{21}} \delta^{\tau-1}\right]. \quad (1)$$

Of course, when we compute $\pi_j(s)$ ex ante (i.e., at the beginning of the game), we have to take into account the probabilities of pairs of states of the demands $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ in each period $t \in N \setminus \{1\}$ generated by s.

For each $h \in H_p$, we denote by $\Gamma_{(h)}$ the public subgame of Γ following the non-terminal public history h. It is the union of subgames following the histories whose public components coincide with h. We indicate restrictions of sets and functions defined for Γ to public subgame $\Gamma_{(h)}$ by subscript "(h)."

A public strong perfect equilibrium (henceforth, PSPE) is the solution concept that we apply to Γ. We obtain its definition by application of Rubinstein's [6] definition of strong perfect equilibrium (henceforth, SPE) to Markov strategies. SPE requires that no coalition in no subgame can strictly Pareto improve the vector of payoffs of its members by a deviation. Thus, in two-player games, it is stronger than various concepts of renegotiation-proofness in infinite horizon two-player games that impose restrictions on strategy profiles to which the grand coalition can deviate (see especially [2], [3], and [1]). In PSPE the deviations are restricted to public strategies or profiles of public strategies. Nevertheless, it is well known (see [4]) that no player in no subgame (when the other player uses a Markov strategy) can increase his payoff by a unilateral deviation to a non-Markov strategy if he cannot increase it by a deviation to a Markov strategy. In Γ, payoffs in a subgame starting in period t do not depend on investments into R&D in the preceding periods. (They depend on the states of the demands affected by R&D investments in the preceding periods but not on R&D investments themselves. Both players observe the states of the demands at the beginning of each subgame.) Therefore, the grand coalition cannot strictly Pareto improve the vector of payoffs in any subgame by using a profile of non-Markov strategies if it cannot do so by using a profile of Markov strategies. Thus, each PSPE is also an SPE. We end this section by giving the formal definition of an PSPE tailored to Γ . Note that part (a) implies that each PSPE is also a subgame perfect equilibrium of Γ .

Definition: A strategy profile $s^* \in S$ is a PSPE of Γ if (a) there do not exist $h \in H$, $j \in \{1,2\}$, and $s_j \in S_{j(h)}$ such that (for $i \in \{1,2\} \setminus \{j\}$) $\pi_{j(h)}(s_j, s^*_{i(h)}) > \pi_{j(h)}(s^*_{(h)})$ and (b) there do not exist $h \in H$ and $s \in S_{(h)}$ such that $\pi_{j(h)}(s) > \pi_{j(h)}(s^*_{(h)})$ for each $j \in \{1,2\}$.

3 EXISTENCE OF AN SPE WITH MORAL HAZARD

In this section, we give a sufficient condition for the existence of an SPE of Γ in Markov strategies with moral hazard, i.e. an SPE in Markov strategies in which no firm invests into R&D but both firms rely on bail out by the government in the low states of the demand for their products.

For each $\omega \in \Omega$, we denote by $\wp(\omega)$ the set of output vectors that generate weakly Pareto efficient vector of single period profits gross of investment costs and bail out payments and give both firms strictly positive single period profit. That is, if $y \in \wp(\omega)$ then it gives both firms strictly positive single period profit (gross of investment costs and bail out payments) and there does not exist $\tilde{y} \in \prod_{j=1}^{2} [0, y_j^{\max}]$ such that $P_j\left(\tilde{y}, \omega_j\right)\tilde{y}_j - c_j\left(\tilde{y}_j\right) > P_j(y, \omega_j)y_j - c_j(y_j)$ for each $j \in \{1,2\}$. Note that, with respect to part (iv) of Assumption 1 and part (iii) of Assumption 3, no payoff vector generated by an element of $\wp(\omega)$ is strictly Pareto dominated by a convex combination of payoff vectors generated by elements of $\wp(\omega)$. Our sufficient condition for the existence of an SPE with moral hazard is based on the following two assumptions.

Assumption 4: For each $\omega \in \Omega$, there exists $y^*(\omega) \in \wp(\omega)$ such that

(i) there do not exist $y(\omega) \in \wp(\omega)$ and $(I_1(\omega), I_2(\omega)) \in \{0,1\}^2 \setminus \{(0,0)\}, \omega \in \Omega,$ such that for some $\omega \in \Omega$

$$\sum_{\omega' \in \Omega} \mu_1(\omega_1, \omega_1', I_1)\mu_2(\omega_2, \omega_2', I_2)[P_j(y(\omega'), \omega_j')y_j(\omega') - c_j(y_j(\omega'))] + \mu_1(\omega_1, \omega_{11}, I_1)\mu_2(\omega_2, \omega_{21}, I_2)M$$
$$- b_j I_j \geq P_j(y^*((\omega_{11}, \omega_{21})), \omega_{j1})y_j^*((\omega_{11}, \omega_{21})) - c_j(y_j^*((\omega_{11}, \omega_{21}))) + M = v_j^*, \forall j \in \{1,2\}, \tag{2}$$

(ii) for each $j \in \{1,2\}$ (with $i \in \{1,2\} \setminus \{j\}$),

$$\mu_j(\omega_{j1}, \omega_{j1}, 1_j)[P_j(y^*((\omega_{11}, \omega_{21})), \omega_{j1})y_j^*((\omega_{11}, \omega_{21})) - c_j(y_j^*((\omega_{11}, \omega_{21})))]$$
$$+ (1 - \mu_j(\omega_{j1}, \omega_{j1}, 1))[P_j(y^*((\omega_{j2}, \omega_{i1})), \omega_{j2})y_j^*((\omega_{j2}, \omega_{i1})) - c_j(y_j^*((\omega_{j2}, \omega_{i1})))_j] - b_j > v_j^*. \tag{3}$$

Assumption 5: For each $j \in \{1,2\}$(with $i \in \{1,2\} \setminus \{j\}$) and for each $\omega_j \in \Omega_j$, there exists $y^{(j)}(\omega_j)$ such that

$$y_i^{(j)}(\omega_j) = 0, \tag{4}$$

$$v_j^{(j)}(\omega_j) = \max\{P_j((y_j, 0), \omega_j)y_j - c_j(y_j) | y_j \in (0, y_j^{\max})\} > v_j^*, \tag{5}$$

$$\mu_j(\omega_{j1}, \omega_{j2}, 1)v_j^{(j)}(\omega_{j2}) + (1 - \mu_j(\omega_{j1}, \omega_{j2}, 1))v_j^{(j)}(\omega_{j1}) - b_j > v_j^{(j)}(\omega_{j1}), \tag{6}$$

$$\mu_j(\omega_{j1}, \omega_{j2}, 1)v_j^{(j)}(\omega_{j2}) + (1 - \mu_j(\omega_{j1}, \omega_{j2}, 1))(v_j^{(j)}(\omega_{j1}) + M) - b_j > v_j^{(j)}(\omega_{j1}) + M , \tag{7}$$

$$\max\{P_i((y_j^{(j)}(\omega_{j1}), y_i), \omega_{i2})y_i - c_i(y_i) | y_i \in (0, y_i^{\max})\} < v_i^*, \tag{8}$$

$$\max\{P_i((y_j^{(j)}(\omega_{j1}), y_i), \omega_{i1})y_i - c_i(y_i) | y_i \in (0, y_i^{\max})\} + M < v_i^*, \tag{9}$$

For each $j \in \{1,2\}$, let $v_j^{\max} = v_j^{(j)}(\omega_{j2})$. That is (taking into account (6)), $v_j^{\max}$ is the maximal possible single period profit of firm j.

Proposition: *There exists $\underline{\delta} \in (0,1)$ such that for each $\delta \in [\underline{\delta},1)$, Γ has an SPE in Markov strategies in which (along the equilibrium path) no firm invests into R&D.*

Proof. Preliminaries. For each $j \in \{1,2\}$ and $T \in N$ consider inequality

$$(1-\delta)v_j^{\max} + \delta(1-\delta^T)M + (\delta^{T-1} - \delta)v_j^* \le 0. \tag{10}$$

For the limit case $\delta = 1$, (10) holds as equality. Differentiating its left hand side with respect to δ, evaluating the derivative at $\delta = 1$, and requiring that it is strictly positive, we find that the left hand side of (10) is strictly increasing in δ at $\delta = 1$ if and only if $T > v_j^{\max}(v_j^* - M)^{-1}$. For each $j \in \{1,2\}$ we set

$$T_j = \min\left\{n \in N \middle| n > v_j^{\max}(v_j^* - M)^{-1}\right\}. \tag{11}$$

Then there exists $\delta_j \in (0,1)$ such that (10) holds (with T_j determined in (11)) for each $\delta \in [\delta_j,1)$ Further, for each $j \in \{1,2\}$ and T_j set in (11) (with $i \in \{1,2\} \setminus \{j\}$) consider inequalities

$$\delta\left[\mu_j(\omega_{j1},\omega_{j2},1)v_j^{(j)}(\omega_{j2}) + (1-\mu_j(\omega_{j1},\omega_{j2},1))v_j^{(j)}(\omega_{j1})\right] - b_j \ge \delta v_j^{(j)}(\omega_{j1}), \tag{12}$$

$$\delta\left[\mu_j(\omega_{j1},\omega_{j2},1)v_j^{(j)}(\omega_{j2}) + (1-\mu_j(\omega_{j1},\omega_{j2},1))(v_j^{(j)}(\omega_{j1}) + M)\right] - b_j \ge \delta\left[v_j^{(j)}(\omega_{j1}) + M\right], \tag{13}$$

$$\max\left\{P_i\left(\left(y_j^{(j)}(\omega_{j1}), y_i\right), \omega_{i2}\right)y_i - c_i(y_i) \middle| y_i \in (0, y_i^{\max})\right\} \le -(1-\delta^{T_i})c_j(0) + \delta^{T_i}v_i^*, \tag{14}$$

$$\max\left\{P_i\left(\left(y_j^{(j)}(\omega_{j1}), y_i\right), \omega_{i1}\right)y_i - c_i(y_i) \middle| y_i \in (0, y_i^{\max})\right\} + M \le -(1-\delta^{T_i})c_j(0) + \delta^{T_i}v_i^*. \tag{15}$$

Using (6)-(9), for the limit case $\delta = 1$, (12)-(15) are satisfied as strict inequalities. Therefore, there exists $\tilde{\delta}_j \in (0,1)$ such that (12)-(15) are satisfied for each $\delta \in \left[\tilde{\delta}_j, 1\right)$. It follows from part (i) of Assumption 4 that there exists $\delta'' \in (0,1)$ such that for each $\delta \in [\delta'',1)$, the claim of part (i) of Assumption 4 continues to hold when we multiply the expression in its first row of (2) and the expression on its right hand side by δ. Further, there exists $\delta' \in (0,1)$ such that for each $\delta \in [\delta',1)$, if profit vector v is generated by an element of $\wp((\omega_{11},\omega_{21}))$, then it is not strictly Pareto dominated by a convex combination of feasible single period profit vectors w and $v' \ne v$, where v' is generated by an element of $\wp((\omega_{11},\omega_{21}))$ and the coefficient of the convex combination assigned to w is no higher than $1-\delta$. We set $\underline{\delta} = \max\left\{\delta_1, \delta_2, \tilde{\delta}_1, \tilde{\delta}_2, \delta', \delta''\right\}$.

Description of strategy profile $s^* \in S$. After the empty history, s^* prescribes the output vector $y^*(\omega(1))$ and no investment into R&D by any firm. Next consider $(h,\omega) \in H_p \times \Omega$ such that we have already defined the prescriptions of s^* for each $(h',\omega') \in H_p \times \Omega$, where h' is strictly contained in h. If h does not contain a unilateral deviation from the prescriptions of s^* or the last unilateral deviation was committed by firm i more than T_i periods ago, s^* assigns the output vector $y^*(\omega)$ and no investment into R&D by any firm to (h,ω). If firm i was the last one that unilaterally deviated from the prescriptions of s^* and it did so T_i or less periods ago, s^* assigns the output vector $y^{(j)}(\omega_j)$ (where $j \in \{1,2\}, j \ne i$), the investment into R&D by firm j and no investment into R&D by firm i to (h,ω).

Strategy profile s^* *is an SPE for* $\delta \in [\underline{\delta},1)$. Since Γ is a game continuous at infinity, the single deviation principle applies to unilateral deviations. Non-profitability of unilateral deviations follows from (10), (14), and (15). (Note that firm $j \in \{1,2\}$ as the punisher earns during the punishment the single period profit exceeding v_j^*. Thus, (10) applies also to the punishment of a unilateral deviation by the punisher. Inequalities (14) and (15) imply that a unilateral deviation by the punished firm can be punished by restarting of the punishment.) Weak Pareto efficiency of the payoff vector generated by s^* in each public subgame follows from Assumption 4 (and the comment concerning (2) in the preliminaries part of the proof), Assumption 5, (12)-(13), and the comment concerning δ' in the preliminaries part of the proof (taking into account part (ii) of Assumption 2).

Q.E.D.

4 CONCLUSIONS

In the present paper, we have demonstrated by a simple model that expectations of bail out by the government can induce a collective moral hazard by the firms – they can refrain from investment into R&D aimed at product innovation and even force one another to such refraining. Punishments are not only based on credible threats but they are also immune to renegotiations by all firms (i.e., immune to temptation to forgive a unilateral deviation). Possibility of this type of collective moral hazard should be taken into account by economic policy makers.

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AUTOMATED SELECTION OF APPROPRIATE TIME-SERIES MODEL

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Abstract
This paper aims to discuss the automated time series analysis. It gives many freeware econometric tools helping users to analyze time series. The aim of such tools is to create some econometric model and inform user about the basic parameters of created model. This paper is focused on Box-Jenkins methodology and the main aim is propose an approach helping user to select the right time-series model to estimate the future values. As in most cases is not possible to select only one model, the offer of more suitable models is also discussed.

Keywords: *time series, Matlab, ARIMA models*

JEL Classification: C44
AMS Classification: 90C15

1 INTRODUCTION

The time series analysis and modelling of time series is very often demanded. The reason is very simple and in most cases it is the need to predict future values of time series. The answer why the prediction is needed is quite clear. Almost everybody wants to know something about the future progress, about the future opportunities. Econometric theory gives answer to this and the existence of econometric models and their usefulness is almost doubtless. But as always nothing is fully ideal. There is high probability that the right model exists, but how to choose them? How to indicate the right one?

Modelling of time series gives two main approaches to analyse. The selection is between the deterministic and stochastic approach. As time series analysis methodology used by authors of this paper is selected the Box-Jenkins methodology representing the stochastic approach.

In 1970 Box and Jenkins made autoregressive integrated moving average (ARIMA) models very popular by proposing a model building methodology with three steps – model identification, estimation of parameters and diagnostic checking (Box and Jenkins, 1968 and Box, Jenkins and MacGrego, 1974) and using obtained model for forecasting. Unfortunately, building an ARIMA model is often a difficult task for users because it requires good statistical practice, knowledge in the field of application and very specialized user-friendly software for time series modelling. There exist a lot of freeware or shareware econometric tools helping users to analyze and model time series. The main aim of such tools is to create econometric model specified by user and inform user about the basic parameters of created model. In many cases the user has to make the decision which model to select and what more, the user is responsible for model verification. Because of using this methodology the model selection is focused on suggestion of some approach to select the right model representing the concrete (ARIMA) process.

Another problem relates with the fact that the number of time series to be analysed is often large and so Box-Jenkins methodology requires both experience and time for successful modelling.There are reasons why we discuss the problem of automated time series analysis. But we are not the first. Also Hoglund and Ostermark (1991) presented study about automatic ARIMA modelling. Unfortunately, authors worked with computer program ANSECH (Mélard, 1982) that is not so popular as the others. Tshman and Leach (1991) published the article about automatic forecast software but in this work was presented automatic forecasting and not entire modelling process with analysis.

Our aim of this paper is to propose an interactive application to enable user not only automated selection of time series model but also to inform user about everything important that has been done. The application aim is also to enable user to suggest his model that was not suggested by our application. The model parameters have to be estimated. In terms of parameter estimation would be used only the implemented methods in existing software (Matlab). It is not the aim of this paper to focus on methods of parameter estimations.

There already exists a lot papers talking about the theory of ARIMA processes, such as Emmenegger (1996) or Makridakis and Hibon (1997), and about the right models for something like typical or ideal situations but reality seems quite different. In theory is described the ideal process but sometimes it is quite difficult to apply it to the real data. Papers presenting some approaches to these examples from real life which do not fit the model as described in theory are rare. Almost always there is some appeal on analytic decision based on his or her experience. Situation when the data do not fit the model exactly is very common and without human interaction is very hard to select right model to be estimated. The reason of writing paper trying to automate the ARIMA model selection is simple because the question of automated selection is not closed yet.

2 ARIMA SELECTION IN MATLAB

In this part of the paper we focused on the selecting criteria for the ARIMA model. The problem of automated model selection is nothing new, but our asset we see in such discussion connected with Matlab and in range of planned use. This paper does not aim to give some genial and cure-all approach. The aim is to inveigle the reader to his own work and self-evident to show and discuss some example of such approach.

This paper deals with one of not very typical tool for construction of time series models. For model selection, estimation and verification is used high-level computing language Matlab. The reason for using Matlab is implemented suitable econometric tool for parameter estimation helping us in our decisions how to select the right fitting ARIMA model. Matlab is often used for time series modelling, e.g. in Kugiumtzis and Tsimpiris (2010) or in Peng and Aston (2011). The reason why we have selected Matlab is to show and emphasize the power of such tool commonly used at technical universities where is available for students. Such tool gives the user wide range of possibilities to assess the model. The strength of tool as Matlab and aim of this paper shows why not to use typical econometric software. Using typical econometric software is nothing bad and in practise it is also seen as a best way. But if the user wants to do more with models and wants to try some own improvement in the common econometric software is not permitted to do some changes. Here we see the biggest advantage of tool, such as Matlab, to our purposes. The implementation of used functions to evaluate basic characteristics of time series is available to user. So not only to program own functions in Matlab but also improvement of existing is possible.

2.1 Two modes

Our suggested application to enable automated model selection is working in two modes. One is set for basic user and second for advanced user. The aim is to extend the number of possible users and not to discriminate the beginners and not to bore the advanced users.

The main difference between the two types of users is that basic user is not informed about all the results of partial processes. As the interactive user is welcome, such user in advanced mode can select whether he or she wants to step in at some checkpoints or not. In case of stepping in there will be as output the selected model with the detailed characteristics for the selection. The user will not be able to refuse some suggested model because the evaluation of suggested model is important to assessment of successfulness. To interactive user facilitate the application to suggest some own ARIMA model not suggested by our application. When user is not interested in this selection program will continue and give all the steps as output at the end.

So after one or more models have been selected to be estimated follows the model verification. It is not needed to show the basic user all models but only models that satisfy our criteria. The reason for our approach to not to show some models is, that some the models that will not passed the verification are useless to be discussed in next. This approach goes with the problem of that

no model has reached criteria. This case our application takes special care and there is created some report to inform about the state of art. The user of application will be informed about the result that no model was found, but will be able to see what models where suggested and why the verification failed. In case of advanced user is the information always complete.
The aim of both user categories is to collect data and evaluate the successfulness of application. To completely accomplish the aim of developed application for automated selection is needed to select how to quantify the results of estimated models. The reason of such approach to our software is that we want to have some feedback from advanced users. We are conscious that every case of real data time series analysis is unique process and so the collecting of data we see as the best way how to improve our application.

2.2 Two phases

The automated model selection we suggest to divide into two phases. The first phase includes model identification. The identification is based on automated approach which is here suggested. The second phase is based on model verification. For estimation of parameters will be used Matlab functions.
To identify the degree of differencing is needed to study the autocorrelation function (ACF). Here is the possible automated approach very significant. It is talked about the unit roots and existence of them can be simply identified form ACF values. To decide about the unit roots existence is needed to have the rule saying when the ACF values are closed to one, the decision is based on statistic approach and the differencing is done when there is more than 90% probability that the first values are closed to one. In case of differencing there is made back control if the differenced time-series really fulfil the stationary. If the results are in this phase not significant there is implemented also the possibility of second differencing. In case of failure of differencing the application tries to make logarithmic transformation. Not to overdifference the time series is the increase of standard deviation also monitored.
After suggestion how to make time series stationary there is another very import decision to select the autoregressive (AR) and moving average (MA) part of ARMA model. In this case are very important ACF and partial autocorrelation function (PACF) values. The theory gives clear description of ACF and PACF in case of typical AR, MA or mixed ARMA processes. To decide about the autoregressive or moving average part of process is needed to research the values on its statistical important and compare them with typical models. Because Matlab provides implemented functions to evaluate ACF and PACF values there is no need to define special functions. The decision how to interpret the values of ACF and PACF is theme of our research and makes the engine of our application. Our suggestion is based on fitting real time series. The end of this phase is the selection of model to evaluate and test. Why do this and not so only to evaluate all common used models, evaluate them and then compare? The reason is simple, to evaluation of model is much more demanding then the right approach to model selection. With the term right approach we mean the approach which would not be more demanding then evaluation of different models.
Nothing wrong is on selecting more models to be evaluated and at this place is our program coming into second phase of use. In this phase will the models be constructed and consecutively verified. The assessment of models is very important and the question is what the right criteria? Verification has three basic steps. Autocorrelation of random element, normal distribution and statistical significant of estimated parameters. The aim is to decide what model has passed the validation criteria. Ideal case of every three criteria fulfilled is rather rare. For basic user we work with 10% probability that the test has not confirmed our expectation. In case of advanced user he is selecting the probability of failing the test. Advanced user can select the probability of failure for each test separately. As the tests are used the commonly used and recommended tests implemented in Matlab. The suggested application deals with ARCH test to ensure about constant standard deviation, to exclude the autocorrelation is used Ljung–Box Q test and research of normal distribution procures the Kolmogorov-Smirnov test. The question

of significant parameters is solved by Matlab when estimating model parameters because the estimation result includes *t*-values which are interpreted by our application at this point.
If after verification there are more models passing verification there are applied criteria designated to this situation. The mainly used are Akaike information criterion (AIC) and Bayesian information criterion (BIC).

3 CONCLUSION

In this paper was discussed the automated time series analysis. Paper is focused on Box-Jenkins methodology and proposes an approach helping user to select the right time-series model to estimate the future values. To decide about stationary time-series is primary used the 90% probability and attention is paid also to standard deviation. Selecting right ARMA model is based on ACF and PACF values and comparing them with the values for typical models known from theory. Model verification is default set to 90% probability of accomplishment criteria enabling user to determine own probabilities for each verification test. Criteria to select one of more suitable models are also mentioned.
The aim of our proposed application tool is also to serve as tool for students and to collect data from them. The term collect sounds at first sight not very good, but the aim is not to select any data connected with user or their analysed data, the data to collect means only the data about successfulness of our application. This data can be in ideal case collected at selected schools or organizations. The goal of selecting data is to quantify the successfulness of our application in real use and consecutively to improve our application based on the results of considerable amount of real data. The feedback will be not only reports comparing models suggested by our application but also application and user selection. If the program will be used in special courses at selected universities, the users will be advanced and the results relevant. The assessment of collected data is the best way to future improvement of our application. The reporting of errors can serve not only to quantifying the application successfulness but also, when using as tool for students, to assess what mistakes are students making. The problem also may be they are using wrong data to be fitted on ARIMA model.

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RELATIONSHIPS BETWEEN GENERAL INDEX AND ITS SECTORAL INDICES: A CASE STUDY FOR SELECTED EUROPEAN INDICES

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Abstract
This paper analyses the relationships between the general European indices (French CAC 40, Portuguese PSI 20, Dutch AEX, Belgian BEL 20) and their sectoral indices published on the NYSE (New York Stock Exchange) Euronext webpage. The whole analysis was based on daily data for the period March 1, 2010 – February 24, 2012 encompassing 515 observations. First the existence of the unit root was tested using the ADF (Augmented Dickey-Fuller) unit root test, followed by the calculation of correlation coefficients, testing of the long-run relationships using the Engle-Granger cointegration test and finally testing the short-run relationships using the Granger causality concept.

Keywords: *general index, sectoral indices, ADF test, correlation, Engle-Granger cointegration test, Granger causality, short-run and long-run relationships*

JEL Classification: C22, C58
AMS Classification: 91G70

1 INTRODUCTION

It is commonly known that the stock exchange provides a platform for trading of stocks, bonds and other securities. The securities markets have had quite a rich history which goes back to the ancient world. Today there exist plenty of stock exchanges around the world. One of the major stock exchanges is the NYSE Euronext, which was created by the merger of NYSE (New York Stock Exchange) Group and Euronext in 2007 and operates the world's leading and most liquid exchange group. It is fast growing and nowadays „the NYSE Euronext equities marketplaces represent one-third of equities trading worldwide."[1] It provides facilities for offering derivatives on commodities, equities, bonds, interest rates, indices, etc. both in the USA and Europe. Since this paper deals with the analysis of stock indices, in further text we will concentrate on this area. The analysis of the general stock indices has been attractive for quite a long time since inter alia there exist the relationship to the economical situation of the corresponding country. Benjelloun and Squalli (2008) argue that it is not so unambiguously true that the general indices reflect the development and behaviour of equity markets and present wide range of studies dealing especially with the performance of emerging markets to document their statement. They accent the necessity to use the sectoral indices in order to reflect the performance of individual sectors more properly. Their analysis of Middle Eastern equity markets confirms that the general indices mask the performance of individual sectors. Waściński et al. (2009) who analysed the relationship between sectoralindices and the WIG showed that in the short-run the development of the WIG had a significant impact on the changes of sectoral indices, while in the long-run such a relationship was not confirmed. Another good reason why to use the sectoral analysis is presented e. g. by Ramkumar et al. (2012) who argue that the studying of the development in individual sectors is useful for investors in order to invest in the stocks of the most promising sector and to achieve better returns. The aim of this paper is to investigate the relationships between the general index and its sectoral indices for a group of four NYSE Euronext markets (France, Portugal, Netherland and Belgium).

[1] http://corporate.nyx.com/en/who-we-are/history

The whole paper is organized as follows: the second section provides a description of the data used in the analysis. The third section discusses the methodology and empirical results. The paper ends with some concluding remarks in the last section.

2 DATA

Daily closing values of the following NYSE Euronext general indices: French CAC 40, Portuguese PSI 20, Dutch AEX and Belgian BEL 20 and various sectoral indices (Basic Materials, Consumer Goods, Consumer Services, Financials, Health Care, Industrials, Oil&Gas, Telecomunications, Technology, Utilities) were used for analysis[2]. The whole analysis was done using the daily data for the period March 1, 2010 – February 24, 2012 encompassing 515 observations. The source of data is the NYSE Euronext webpage[3], the whole analysis was carried out in the econometrical software EViews 5.1 on the logarithmic transformation of the individual index series.

Since the typical feature of time series measured in levels (i.e. time series of individual stock indices) is the non-stationarity, the analysis is usually done on time series of the first differences (i.e. time series of returns). If we denote the closing value of the stock index in time t as P_t, the expected stock return r_t can be calculated using the following formula: $r_t = \ln(P_t / P_{t-1}) * 100\%$.

Descriptive statistics of individual index and return series were calculated and can be provided by the author upon request. Concerning the standard deviation in the stock index series it was much higher than in the return series. While in case of stock index series it varied between 4,61 % - LTEL (B)[4] and 42,66 % - LFIN (P), in case of returns the values were considerably lower and the range was quite narrow: from 1 % - DLCON_SER (NL) to 2,45 % - DLFIN (P). Furthermore we compared the character of the distribution of individual index and return series with the normal distribution using the calculated Jarque-Bera test statistics based on skewness and kurtosis. While the skewness for stock index series was in majority cases negative, for returns it was both positive and negative. The kurtosis values were for stock index series lower than 3 (the only exception was LCON_GO (NL)), for returns much higher (in case of DLHC (NL) more than 130) indicating leptokurtic distributions. The normality hypothesis based on Jarque-Bera test could also be rejected in all analysed cases.

3 METHODOLOGY AND EMPIRICAL RESULTS

As it was already mentioned above, the individual stock index series are usually non-stationary, but their first differences, i.e. return series, are usually stationary. To test the time series on non-stationarity, i.e. on the existence of the unit root, is an important issue in order to carry out correct further econometric analysis. There are many unit root tests (see e.g. Enders (1995), Franses and van Dijk (2000)) which are by some authors in case of financial time series analysis used also to test the efficient markets hypothesis[5]. While Brooks (2008) asserts that the efficient markets hypothesis implies that the stock index (or the natural logarithm of stock index) should follow a random walk or random walk with a drift (i.e. to be I(1)), so that its first differences (returns) are unpredictable, there are another authors who came to the conclusion that from the testing of the random walk hypothesis could be nothing concluded about the efficiency of the corresponding market (see e.g. Urrutia (1995), Benjelloun and Squalli (2008)). We tested the stationarity of all the analysed time series (both indices and returns) using the ADF test

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[2] The number of analysed sectors was in case of individual countries not identical and it was dependent on the data presented on the following webpage: http://www.euronext.com/trader/indicescomposition/allindices_sectorialindices-1909-EN.html (valid on February 27, 2012)

[3] https://indices.nyx.com/directory/equity-indices (valid on February 27, 2012)

[4] Prefix „L“ denotes in the whole text the natural logarithm and prefix „D“ the first difference. The abbreviation in the brackets indicates the concrete country: F – France, P – Portugal, NL – Netherland, B – Belgium.

[5] For more information about market efficiency and (im)possibilities of testing it see e.g. Campbell et al. (1997).

following the testing scheme of Enders (1995).[6] Since for indices (levels) it was not possible to reject the existence of unit root on the significance level 1 %, for the returns we were able to reject the hypothesis using the ADF test both with constant and trend. All the index series are therefore non-stationary – I(1) and the return series stationary – I(0).

Furthermore the relationships between the general indices (LCAC 40, LPSI 20, LAEX and LBEL 20) and corresponding sectoral indices as well as relationships between corresponding return series were investigated using the correlation analysis, Engle-Granger cointegration concept and Granger causality test.

Correlation coefficients both for indices and returns are in Appendix 1. The strongest relationships (correlation coefficient more than 0,9) were indentified between LCAC 40 and its four sectoral indices, LAEX and its four sectoral indices and LBEL 20 and its three sectoral indices. In case of LPSI 20 and its sectoral indices the correlation coefficients were very low indicating weak or no relationships between analysed index series. Concerning the return series it can be concluded that there exist a strong relationship for the Euronext Paris (correlation coefficients with one exception were higher than 0,8), not so strong for Euronext Amsterdam and Euronext Brussels. The situation for Euronext Lisbon was a little bit different, since the correlation coefficients don't exceed 0,5 with one case of no relationship (DLCON_GO).

Since the time series of individual indices are non-stationary, we can employ the cointegration concept to test the long-run relationships between general indices and corresponding sectoral indices. We used the Engle-Granger cointegration method, i.e. we estimated the following equation:

$$y_t = \varphi_0 + \varphi_1 x_t + \varepsilon_t, \qquad (1)$$

where y_t denotes the logarithms of individual sectoral indices, x_t denotes the logarithm of the corresponding general index, φ_0 and φ_1 are unknown parameters and ε_t is an error term. On the residuals from equation (1) we applied the ADF test both without trend and constant (see Enders (1995)). The rejection of the null hypothesis that the residual sequence from equation (1) contains a unit root implies that the variables y_t and x_t are cointegrated (i.e. confirmation of the long-run relationship between these variables). Otherwise there is no long-run relationship between the analysed variables. From the Appendix 2 it seems to be clear that it was not possible to reject the unit root hypothesis on the significance level 1 % nor on the significance level 5 %, so on these significance levels no long-run relationship was confirmed. Only in two cases LBAS_MA (F) and LTEL (B) the residuals were identified to be stationary on the significance level 10 % indicating the possibility of the long-run relationship.

The short-run relationships were tested using the Granger causality concept. Since the analysed time series were non-stationary, the Granger causality test had to be applied on returns (i.e. first differences). The inclusion of the error correction term e_{t-1} is dependent on the fact whether the existence of cointegration between the analysed series was confirmed or not. Using the above mentioned denotation of variables it can be said, that the return series Dx_t Granger-causes return series Dy_t if Dy_t can be predicted better by using past values of Dx_t than by using only the historical values of the Dy_t. If this doesn't hold, we can say that there is no short-run relationship between the analysed series. Since we tested only the unidirectional causality from general indices to corresponding sectoral indices, the corresponding model is as follows:

$$Dy_t = \alpha_1 + \phi_1 e_{t-1} + \sum_{j=1}^{k} \alpha_j Dx_{t-j} + \sum_{j=1}^{k} \beta_j Dy_{t-j} + \xi_t, \qquad (2)$$

where e_{t-1} denotes the deviation from the long-run equilibrium (see equation (1)), the variable ξ_t has a character of the white noise and parameter ϕ_1 measures the speed of adjustment. Usual Wald F statistic could be used in order to test the Granger causality (in this paper we tested the

[6] The results can be provided by the author upon request.

existence only of the unidirectional causality, i.e. if Dx_t Granger-causes return series Dy_t). The results of the Granger test (without consideration of the cointegration relationship between LCAC40 and LBAS_MA, LBEL20 and LTEL, i.e. without inclusion of the error correction term e_{t-1} into (2)) are summarized in Appendix 3. The analysis was done for different number of lags ranging from 1 to 5 lags. The significant short-run relationships were confirmed in several cases: DLCAC40 and DILNDU, DLAEX and DLCON_SER, DLAEX and DLINDU, DLBEL20 and DLCON_SER, DLBEL20 and DLTECH (for all the lags 1 – 5); DLBEL20 and DLBAS_MA, DLBEL20 and DLINDU (for the lags 2 – 5); DLCAC40 and DLUTI, DLPSI20 and DLUTI, DLAEX and DLBAS_MA (for the three successive lags). Taking into account the two above mentioned cointegration relationships, model (2) with e_{t-1} included was estimated using 1 – 5 lags. In case of DLCAC40 and DLBAS_MA the parameter ϕ_1 was not statistically significant and it was not possible to reject the null hypothesis that DLCAC40 doesn't Granger-causes DLBAS_MA. Another results were received for DLBEL20 and DLTEL, where the speed of adjustment varied between 0,050 (1 lag) and 0,065 (5 lags) and was highly statistically significant (significance level 1 %). Also in this case the hypothesis that DLBEL20 doesn't Granger-cause DLTEL could not be rejected.

4 CONCLUSION

The paper analysed the relationships of selected Euronext general indices to their sectoral indices. Firstly we tested the existence of unit root using the ADF test on all series (both general indices and sectoral indices). Similarly as Waściński et al. (2009) we failed to reject the hypothesis about the existence of the unit root in all general and sectoral index series, but the first differences (returns) were stationary. Since in this analysis there was no difference between results received for general and individual sectoral indices, the results do not support those of Benjelloun and Squalli (2008) that investors and policy makers should avoid to use general indices in their analyses. Concerning the correlation coefficients and the results of the Granger causality tests, there are some interesting remarks to be accented. For majority of pairs with confirmed short-run relationships the correlation coefficients were high, but on the other hand there are some pairs with very high correlation coefficients with no short-run relationships (compare Appendices 1 and 3). The long-run relationships based on Engle-Granger cointegration test were confirmed only in two cases on the significance level 10 %. It is also clear that based on this study no general conclusion can be made, since the existence of the short-run and long-run relationships of the general index to its sectoral indices is strongly dependent on the concrete analysed pair of indices.

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Appendix 1 Correlation coefficients of general indices with sectoral indices

Euronext Paris

	LBAS_MA	LCON_GO	LCON_SER	LFIN	LHC	LINDU	LOIG	LTECH	LTEL	LUTI
levels	0,9635	0,2431	0,9483	0,9642	0,1882	0,9056	0,7775	0,8632	0,8398	0,8902
returns	0,9127	0,9146	0,9446	0,9224	0,7997	0,9523	0,9120	0,8369	0,8298	0,8799

Euronext Lisbon

	LBAS_MA	LCON_GO	LCON_SER	LFIN	LINDU	LTEL	LUTI
levels	0,0405	0,3605	0,1603	-0,1222	0,0277	-0,1140	0,1950
returns	0,3362	0,0648	0,4283	0,4229	0,4954	0,4135	0,4223

EuronextAmsterdam

	LBAS_MA	LCON_GO	LCON_SER	LFIN	LHC	LINDU	LOIG	LTECH	LTEL
levels	0,9244	0,8484	0,9171	0,9393	0,7535	0,9440	0,2013	0,6142	0,6516
returns	0,8948	0,8488	0,8446	0,9249	0,3948	0,8803	0,7876	0,7601	0,5056

Euronext Brussels

	LBAS_MA	LCON_GO	LCON_SER	LFIN	LHC	LINDU	LTECH	LTEL	LUTI
levels	0,4509	-0,1574	0,3164	0,9481	-0,0437	0,9442	0,3717	0,7987	0,9277
returns	0,8615	0,6748	0,7315	0,9366	0,7148	0,8358	0,5726	0,6907	0,8379

Appendix 2 Engle-Granger cointegration test – residuals ADF test statistics

Dependent variable →	LBAS_MA	LCON_GO	LCON_SER	LFIN	LHC	LINDU	LOIG	LTECH	LTEL	LUTI
Euronext Paris	-3,13098*	-2,06994	-1,83409	-2,44974	-2,23486	-1,61948	-1,02131	-1,9348	-0,96715	-0,75505
Euronext Lisbon	-1,69331	-3,03743	-2,06704	-0,6602	-	-0,89935	-	-	-0,72548	-2,52659
Euronext Amsterdam	-2,48018	-3,02735	-2,33871	-2,08952	-2,45456	-2,75117	-1,78739	-0,87917	-0,78831	-
Euronext Brussels	-1,30949	-2,01595	-3,00466	-1,92369	-2,23975	-3,05091	-	-1,40337	-3,31677*	-2,55025

*Note: Symbols ***, **, * denote in all appendices the rejection of the null hypothesis on the 1 %, 5 %, and 10 % significance level, respectively.*

Appendix 3 Granger causality test (H_0: General index returns does not Granger cause sectoral index returns)

Euronext Paris	F-statistics				
	Lags: 1	Lags: 2	Lags: 3	Lags: 4	Lags: 5
DLBAS_MA	0,0287	1,5446	1,7589	1,2388	1,0376
DLCON_GO	0,0293	0,0893	0,0869	0,1224	0,112
DLCON_SER	3,8804**	2,025	1,6678	1,2514	1,0675
DLFIN	0,0668	0,5805	1,3417	1,3821	1,1246
DLHC	0	1,0847	0,7414	1,0077	0,8323
DLINDU	4,2805**	4,2494**	2,8380**	2,2819*	1,8556*
DLOIG	0,1238	0,9489	0,7283	0,9375	0,9632
DLTECH	0,5272	0,8672	0,7791	0,9892	1,0383
DLTEL	0,0023	0,0185	0,289	0,346	0,4185
DLUTI	0,6905	3,4399**	2,2622*	2,0008*	1,6479

Euronext Lisbon	F-statistics				
	Lags: 1	Lags: 2	Lags: 3	Lags: 4	Lags: 5
DLBAS_MA	2,7628*	1,9267	1,2929	0,9846	1,7874
DLCON_GO	3,4650*	2,8785*	1,9118	1,2653	1,3816
DLCON_SER	1,2978	1,7646	1,2051	0,9210	0,7748
DLFIN	0,4171	0,1375	0,3621	0,7461	0,5929
DLINDU	2,2833	1,1337	0,8412	1,2223	1,0384
DLTEL	0,7994	0,3741	0,2595	0,2046	0,2563
DLUTI	5,5166**	3,5618**	2,4934*	1,8485	1,4916

Euronext Amsterdam	F-stastistics				
	Lags: 1	Lags: 2	Lags: 3	Lags: 4	Lags: 5
DLBAS_MA	0,2579	2,6568*	2,3116*	2,2241*	1,7889
DLCON_GO	0,2134	0,7560	0,8691	0,6923	0,6442
DLCON_SER	9,3063***	4,8284***	3,2660**	2,5504**	2,1158*
DLFIN	1,2918	0,6601	0,5325	0,5528	0,5566
DLHC	2,5155	1,1845	0,8099	0,6098	0,5093
DLINDU	3,2203*	3,4277**	2,5070*	2,2781*	3,2654***
DLOIG	0,0686	0,0370	0,3193	0,3748	0,5706
DLTECH	0,1174	0,2553	1,3916	1,0361	1,4629
DLTEL	0,0332	0,8665	1,7047	1,4140	1,1833

Euronext Brussels	F-statistics				
	Lags: 1	Lags: 2	Lags: 3	Lags: 4	Lags: 5
DLBAS_MA	0,8371	2,7188*	2,1991*	2,6016**	2,1433*
DLCON_GO	2,5107	1,2336	0,6974	1,0890	0,9997
DLCON_SER	9,9216***	4,9193***	3,3073**	2,3808*	2,2213*
DLFIN	0,4418	0,1254	0,1209	0,4651	0,6421
DLHC	1,0270	0,4495	0,3186	0,3369	0,2722
DLINDU	1,7715	3,4450**	3,6226**	2,8404**	2,7108**
DLTECH	16,9737***	8,8780***	7,8256***	5,8693***	4,9043***
DLTEL	0,3914	0,8430	1,8950	1,9201	1,6552
DLUTI	0,3177	0,9131	0,6384	1,4068	1,1485

DOUBLE-CRITERION OPTIMALIZATION OF DISTRIBUTIVE SYSTEM STRUCTURE

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Abstract
One of the key tasks which are necessary to be solved in case of distribution of products from places of production to final customers, is a problem of optimization of distributive system. In this problem we decide the method (the way) how to supply the customers. If the supplies are going to proceed through in-process store or directly. Although the issue of optimization of distributive system structure can be actual not only when it is proposed but in any phase of its process. The important phase of optimization process is the choice of suitable optimization criterion. Generally the optimization criterion must fulfill two basic conditions. It must express the real interest of an operator (should take in account the problems which an operator needs to remove) and must be also well quantifiable. In the submitted report there are used two criterions for suggestion of distributive system structure – the costs for maintenance of distributive system are minimized and reliability of delivery in time is maximized. The problem solution is done by methods of mathematician programming. The solution of the model itself is performed in optimization software Xpress-IVE which is the most suitable for this kind of problems.

Keywords: *mathematical programming, reliability of distribution systems, optimization criteria*

JEL Classification: C30
AMS Classification: 90C29

INTRODUCTION

In general, distribution systems deal with the flow of goods between places where goods acquire their final form and places of consumption. At present, apart from the distribution system costs optimization, there is now more and more often a requirement that timely shipment delivery reliability is provided. To handle this issue, there are several different approaches based on the use of optimization models. The article discusses the creation of a mathematical model for designing the structure of a distribution system both with direct supplying of the customer from a primary source and supplying through buffer stocks. The aim of this article is to outline the optimization of two criteria. The first optimality criterion will include the distribution system operation costs that will get minimized while the second criterion will be the reliability of goods delivery to the end consumer that will get maximized.

1 INPUT INFORMATION FOR MATHEMATICAL MODEL DESIGN

To design the model, it is necessary to define the issue, lay down the optimization process goals and then process and evaluate the results obtained.
In implementing the model, a graph where peaks represent the primary source, buffer stocks of goods and end customers that require to be supplied with goods in time (just in time) may be used for better illustration. The edges of this model represent the minimum paths between the primary stocks and end customers on one hand and minimum paths between buffer stocks and end customers on the other hand.
Both direct supplying of the end customer from the primary source and supplying of end customers from buffer stocks is considered in this article. Each buffer stock will have a predefined capacity of goods and operation requirements will be known in respect of each end customer. As far as the primary source is concerned, let us expect its capacity will not be limited. Further input information includes the yearly costs of operation of the buffer stocks.

Vehicles providing the transport of shipment will also be defined. In such a case, two types of vehicles of various capacities will provide the distribution. Operation costs of covering one kilometer will further be defined for both vehicles. Timely shipment delivery reliability values for each path both between the primary source and the end customer and the buffer stock and the end customer must further be defined in the model. The values are obtained based on the operation conditions on the given path. In practice, there may be a situation that an economically advantageous path has a substantial timely shipment delivery probability of failure while on the other hand, there may be a higher timely shipment delivery probability on a path with increased costs. The aim of the model is to create an optimum method of supplying customers from the perspective of both criteria.

The general procedure of the mathematical model creation is given e.g. in [4].

Recapitulation of the input values for the task being solved:

I	Set of localities where building of stocks is considered plus the primary source (primary source will be assigned index 0)	[-]
J	Set of customers	[-]
d_{0i}	Distance from the primary source to the locality $i \in I$	$[km]$
d_{ij}	Distance from the locality $i \in I$ to the end customer $j \in J$	$[km]$
f_i	Annual buffer stock operation costs $i \in I$	$[mu \cdot year^{-1}]$
q_i	Capacity of the buffer stock if built in the locality $i \in I$	$[pieces \cdot year^{-1}]$
k_1	Capacity of the vehicle servicing the primary source – buffer stock relations	$[pieces]$
k_2	Capacity of the vehicle servicing the buffer stocks – end customers relations	$[pieces]$
c_1	Costs of covering one kilometer by a vehicle with the capacity k_1	$[mu \cdot km^{-1}]$
c_2	Costs of covering one kilometer by a vehicle with the capacity k_2	$[mu \cdot km^{-1}]$
e_{0j}	Distance of the primary source and the end customer $j \in J$	$[km]$
h_{0j}	Costs of covering 1 km between the primary source and the end customer $j \in J$ by a vehicle with the capacity k_1	$[mu \cdot km^{-1}]$
p_{0j}	Reliability of timely shipment delivery from the primary source to the end customer $j \in J$	[-]
p_{ij}	Reliability of timely shipment delivery from the primary source to the end customer $j \in J$ from the buffer stock $i \in I$	[-]
b_j	Annual requirement of the end customer $j \in J$	$[pieces \cdot year^{-1}]$

2 MATHEMATICAL MODEL

As stated above, it is necessary to include in the model variables that will define individual decisions. In such a case, it is necessary to make a decision on whether or not a buffer stock will be built in the specified localities. If a buffer stock is to be built, the buffer stock from which the end customer will be supplied needs to be determined or it will be necessary to determine if supplying directly from the primary source will be more economical. This method of decision-making will apply in case we minimise the distribution system operation costs.

To maximise the timely shipment delivery reliability, which is the second criterion of this model, another variable considering the reliability must be introduced in the model. For our needs, a variable representing the minimum reliability of supplying the customer from the specified place (primary source, buffer stock) will be selected. Its value will be maximized as part of the optimization process.

The following variables will therefore be included in the model:

x_{ij}	Bivalent variable modelling a decision whether or not the end customer $j \in J$ will be supplied from the locality $i \in I$	[-]
y_i	Bivalent variable modelling a decision whether or not a buffer stock will be built in the locality $i \in I$	[-]

z_j	Bivalent variable modelling whether or not the end customer j∈J will be supplied from Primary source	[-]
d	Variable representing the minimum timely shipment delivery reliability value	[-]

The input details common with the defined variables will be used to formulate the effectiveness value and the set of restricted conditions. The restricted conditions in the model must:

- Provide supplying of each end customer $j \in J$,
- Assign the maximum achievable timely shipment delivery reliability for supplying of each customer,
- Make sure the sum of requirements of assigned customers does not exceed the capacity of buffer stocks,
- Provide a correct link of the values of variables modelling respective decisions in a task,
- Specify the definition fields of individual variables.

The mathematical model will therefore include two effectiveness values where the first one will represent the total costs connected with the distribution system operation and which we will try to minimise in this model. The second effectiveness function will represent the achieved timely shipment delivery reliability value and which we will try to minimise. The optimization software will then enable the investigator to find optimum values in terms of the operation and reliability costs (considering the priorities of the criteria).
The mathematical model will have the following form:

$$\min f(x,y) = \sum_{j \in J} h_{0j} \cdot e_{0j} \cdot \frac{b_j}{k_1} \cdot z_j + \sum_{i \in I} f_i \cdot y_i + \sum_{i \in I} \sum_{j \in J} (c_1 \cdot d_{0i} \cdot \frac{b_j}{k_1} + c_2 \cdot d_{ij} \cdot \frac{b_j}{k_2}) \cdot x_{ij} \quad (1)$$

$$\max f(d) = d \quad (2)$$

Under the following conditions:

$$\sum_{i \in I} x_{ij} + z_j = 1 \quad \text{for } j \in J \quad (3)$$

$$\sum_{j \in J} b_j \cdot x_{ij} \le q_i \quad \text{for } i \in I \quad (4)$$

$$p_{ij} \cdot x_{ij} + p_{0j} \cdot z_j \ge d \quad \text{for } i \in I; j \in J \quad (5)$$

$$x_{ij} \le y_i \quad \text{for } i \in I; j \in J \quad (6)$$

$$x_{ij} \in \{0,1\} \quad \text{for } i \in I; j \in J \quad (7)$$

$$z_j \in \{0,1\} \quad \text{for } j \in J \quad (8)$$

$$y_i \in \{0,1\} \quad \text{for } i \in I \quad (9)$$

$$d \ge 0 \quad (10)$$

The function (1) is the effectiveness function representing the annual distribution system operation costs that we try to minimise. The function (2) is the effectiveness function representing the minimum timely shipment delivery reliability value (member including the variable d). The group of restricted conditions (3), the number of which corresponds to the number of end customers, will make sure that each end customer is supplied just from one place (either from the primary source or from just one buffer stock). The group of restricted conditions (4), the number of which in the model corresponds to the number of buffer stocks, will make sure the no capacity of any stock will be exceeded. The restricted conditions in the group (5) create relations between the assigning variables and the timely shipment delivery reliability level. The group of restricted conditions (6), the number of which corresponds to the product of end customers and the number of buffer stocks, will make sure that if a buffer stock is not built in the locality, no end customer will be assigned to that locality. Further, it makes sure that if a customer is assigned to a locality suitable for building a buffer stock, a buffer stock will be built in that locality. The groups of restricted conditions (7), (8), (9) and (10) specify the definition fields of variables.

3 COMPUTING EXPERIMENTS

3.1 Input data

To verify the functionality of the model it has chosen a fragment of the real distribution system. The company operates with a central stock in Unhošt, which will represent the primary source. The distribution of goods to the end customers will be realized through four buffer stocks in Cologne (here and after referred to as „Stock 1“), Hradec Králové (here and after referred to as „Stock 2“), Liberec (here and after referred to as „Stock 3“), and Karlovy Vary (here and after referred to as „Stock 4“). From the group of end customers, will be chosen four end customers that have business in Kutná Hora (here and after referred to as „Customer 1“), Jaroměř (here and after referred to as „Customer 2“), Nový Bor (here and after referred to as „Customer 3“), and Horní Slavkov (here and after referred to as „Customer 4“). The input data representing the path between the entities are reported in table.

Table 1 Input data of the proposed example– distances between the entities [km]

	Stock 1	Stock 2	Stock 3	Stock 4	Primary source
Customer 1	15,3	63,2	119	218	117
Customer 2	76	22,6	101	263	162
Customer 3	111	156	43	170	115
Customer 4	215	259	255	19,6	116
Primary source	102	148	144	105	0

Other input data that are needed to be assigned to the model are the annual cost of operation of the stock– this cost is considered with the following values:

- Stock 1: 15 000 money units·year^{-1};
- Stock 2: 30 000 money units·year^{-1};
- Stock 3: 25 000 money units·year^{-1};
- Stock 4: 10 000 money units·year^{-1}.

Other input data – timely shipment delivery probabilities for the utilization of stock in test task are reported in table 2.

Table 2 Input data of the proposed example– timely shipment delivery probabilities [-]

	Stock 1	Stock 2	Stock 3	Stock 4	Primary source
Customer 1	0,9	0,75	0,4	0,7	0,7
Customer 2	0,7	0,45	0,6	0,5	0,8
Customer 3	0,6	0,85	0,7	0,75	0,65
Customer 4	0,6	0,75	0,9	0,9	0,9

For solving the model is also necessary to know the capacity of the vehicles that will transport the requested goods and the costs of covering 1 km of distance. Capacity of the vehicle k_1 is 50 units of goods and the capacity of the vehicle k_2 25 units of goods. The s of the vehicle c_1 is 80 money units·year^{-1} and the operation costs of the vehicle c_2 is 60 money units·year^{-1}.
The last group of input information, that is necessary to introduce into the optimization software, is the capacity of the stock and the requirement of the individual customers. The planning time is 1 year. This information is reported in table 3.

Table 3 Input data of the proposed example— capacity of the stock and customer requirements [$mu \cdot year^{-1}$]

pacity of the stocks [$mu \cdot year^{-1}$]			
Stock Kolín	Stock H.K.	Stock Liberec	Stock K.V.
5500	7000	6000	4000
Annual requirements for end customers [$mu \cdot year^{-1}$]			
Customer 1	Customer 2	Customer 3	Customer 4
400	100	700	850

3.2 Implementation of the optimization calculation and plan experiments

Based on the input conditions above, a mathematical model will be prepared, with the determination of the optimum method of supplying the end customers. Apart from the solution itself, the optimization software also enables to define types of outputs. The program text was defined in a way that after starting the algorithm the solver obtains not only information on the objective function value, summary of end customers assigned to individual buffer stocks in case of directly delivery from the primary stocks.

The resolution of the problem was performed in the optimization software Xpress-IVE.

The proposed two criteria model was solved by one of the basic approaches, which are used in the multi-criteria programming (Černá-Černý: Theory of management and decision in the transportation systems). In this approach, first of all, the sequence of the individual criterion is determined.

After the determination of the sequence of the criteria the problem will be solved as one criterion and the objective function in the solving problem contains one criterion, which is included in the first place. After solving the problem the second model is assembled, again the second model will be with one criterion. The objective function in the second model will contain a criterion, which was in the second place. The achieved value of the objective function from the first model, in the second model, is transformed into a restricted condition that is added to the system of restricted conditions. With the assembled model of the given real system two experiments were realized. The experiments differ in the sequence of criteria.

In the computing experiment n.1 a higher priority is assigned to the total costs of operation in the distribution system in the computing experiment n.2 the minimum reliability of the delivery the consignment in time.

3.3 Evaluation of the progress and results of computing experiment n. 1

After all the input data and the introduction of the restricted conditions and the objective function the first step was made. In the first step the minimum operation costs in the distribution system was founded using the objective function (1) and under the defined conditions (3), (4) and (6) – (9). According with the output data for the objective function was obtained a value of 515 512 money units·year^{-1}.

In the second step the value of minimum reliability for the delivery the consignment in time will be maximized, and it has to be respected that the overall cost of the operation for the distribution system can exceed the value 515 512 money units·year^{-1}. So a model is solved, in which the objective fiction (2) will be minimized respect the conditions (3) – (10) and (11).

$$\sum_{j \in J} h_{0j} \cdot e_{0j} \cdot \frac{b_j}{k_1} \cdot z_j + \sum_{i \in I} f_i \cdot y_i + \sum_{i \in I} \sum_{j \in J} (c_1 \cdot d_{0i} \cdot \frac{b_j}{k_1} + c_2 \cdot d_{ij} \cdot \frac{b_j}{k_2}) \cdot x_{ij} \leq 515\,512 \qquad (11)$$

According with the output data the minimum value of reliability of the consignment in time of 0,65 was achieved by the optimization software.

3.4 Evaluation of the progress and results of computing experiment n. 2

After assigning all the input data and the introduction of the restricted conditions and the objective function the first step is already made. Unlike the computing experiment n.1 in the computing experiment n.2 was maximized the minimum value of the reliability of delivery the consignment in time (without restriction of costs of operation in the distribution system). So the mathematical model was solved and where the objective function was maximized (2) in the compliance of conditions (3) – (10). In this case after the end of the optimized calculation the value of the objective function that was achieved is 0,8.

Here can be seen the visible difference in comparison to the second iteration and this is due to the fact, that we have no limits for example in the financial fund, and we are able to deliver goods to the customers in time with a higher reliability. At present there are many customers, who are willing to pay for a dealer access, only for obtaining goods in time.

In the second step a minimum value of reliability was achieved and transferred to the restricted conditions and under the conditions (3) – (9) and (12)

$$d \geq 0{,}8 \qquad (12)$$

The value of the overall operation costs was minimized in the distribution system and it was defined by function (1). After the end of the computing experiment the value of the objective function achieved is 784 472 money units·year^{-1}. As it can be seen from the results of the experiment, from the requirement of higher reliability for the consignment in time it is expected a higher operation cost in the system. The results of both experiments are summarized in table 4.

Table 4 The genesis of the solution results after each steps

Computing experiment n. 1		Computing experiment n. 2	
Step 1 $\min f(x,y,z)$	Step 2 $\max f(d)$	Step 1 $\max f(d)$	Step 2 $\min f(x,y,z)$
OF:515 512	OF:0,65	OF:0,8	OF:784 472
x(1,1)=1	x(1,1)=1	x(1,1)=1	x(1,1)=1
x(1,2)=1	x(1,2)=1	x(1,2)=0	x(1,2)=0
x(1,3)=0	x(1,3)=0	x(1,3)=0	x(1,3)=0
x(1,4)=0	x(1,4)=0	x(1,4)=0	x(1,4)=0
x(2,1)=0	x(2,1)=0	x(2,1)=0	x(2,1)=0
x(2,2)=0	x(2,2)=0	x(2,2)=0	x(2,2)=0
x(2,3)=0	x(2,3)=0	x(2,3)=1	x(2,3)=1
x(2,4)=0	x(2,4)=0	x(2,4)=0	x(2,4)=0
x(3,1)=0	x(3,1)=0	x(3,1)=0	x(3,1)=0
x(3,2)=0	x(3,2)=0	x(3,2)=0	x(3,2)=0
x(3,3)=0	x(3,3)=0	x(3,3)=0	x(3,3)=0
x(3,4)=0	x(3,4)=0	x(3,4)=0	x(3,4)=0
x(4,4)=0	x(4,4)=0	x(4,4)=0	x(4,4)=0
x(4,4)=0	x(4,4)=0	x(4,4)=0	x(4,4)=0
x(4,4)=0	x(4,4)=0	x(4,4)=0	x(4,4)=0
x(4,4)=1	x(4,4)=1	x(4,4)=0	x(4,4)=1
y(1)=1	y(1)=1	y(1)=1	y(1)=1
y(2)=0	y(2)=0	y(2)=1	y(2)=1
y(3)=0	y(3)=0	y(3)=1	y(3)=0
y(4)=1	y(4)=1	y(4)=1	y(4)=1
z(1)=0	z(1)=0	z(1)=0	z(1)=0
z(2)=0	z(2)=0	z(2)=1	z(2)=1
z(3)=1	z(3)=1	z(3)=0	z(3)=0
z(4)=0	z(4)=0	z(4)=1	z(4)=0

4 CONCLUSION

The article submitted deals with the issue of solving tasks regarding the optimization of the structure of a distribution system with the usage of a two criteria mathematical model. In practice, there are a number of situations where the end customer requires to be supplied in time. On the other side this requirement is in turn reflects the operation costs in the distribution system. If the customer requests higher timely shipment delivery probability, it is necessary a higher cost of operation in the distribution system. The article includes a mathematical model to solve the given task. Its functionality was verified in a real fragment of a real distribution system. The solution was performed using the Xpress – IVE optimization software.

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RELATIONS BETWEEN FISCAL POLICY AND REGIONAL INCOME

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Abstract
The present paper is focused on the analysis of the unstable development of the regions of Slovak economy. Analysis is based on the equations of the budget of the consumers, on the assumption that in economy exist only such system of the taxes which are purely dependent on state that means the fiscal policy has no instrument of pure tax redistribution. Optimal size of the intranational risk sharing resulting from the unstable development of regions is obtained by choosing the tax rates which ensure the minimizing the variance of consumption. On the basis of the received results it is possible to consider, that under minimizing the variance of consumption, the present tax rate in the Slovak economy is fulfilled only for Bratislava's region.

Keywords**:** consumption smoothing, fiscal policy, intranational risk-sharing,

JEL Classification: E21, E24, C44
AMS Classification: 91B42, 91B82

I INTRODUCTION

A fundamental feature of the modern state is to provide risk-sharing arrangement for its citizens, defined as sharing income risk among the inhabitants of the different regions of a state. This can take a variety of forms in practice. Often, it is a product of general welfare and tax-transfer systems. In some federal states, the intranational risk sharing is provided by fiscal mechanisms designed for the horizontal redistribution of the income among subcentral governments. In other state, the intranational risk sharing is product of budgetary transfers from the central government to regional or local governments. Such mechanisms are generally based on equity considerations. Very often both instruments of intranational risk sharing are combined. It is important to notice, that protecting the individual against economic hardships in regions is part of solidarity in society.

The discussion over European monetary integration in the last years has produced numerous empirical studies about intranational insurance. Intranational risk-sharing has an obvious aspect of the intranational economic stabilization. Redistribution of income from prospering regions to regions in distress can help to attenuate asymmetries in the cyclical fluctuations of different regions belonging to the same country (to the same Union or Federation) and produce a more even economic development across all regions (states). This aspect has gained particular attention in the context of the European Monetary Union in the past 30-35 years. Some of the interesting ideas: Delors (1989) talked about a fiscal risks-haring mechanism among the members of the EMU. Sachs and Sala-i-Martin (1991) claim that successful EMU must be vested with instruments for regional redistribution comparable to the existing in the US, his empirical analysis pointed out that the loss of exchange rate channel for adjustment to asymmetric shocks must be compensated by an appropriate fiscal policy.

The economists have approached intranational risk-sharing from two aspects. One strand of literature considers risk-sharing among consumers inhabiting different regions as a special case of consumer smoothing. The other one regards intranational transfer mechanisms as an alternative to flexible exchange rates and their market mechanisms for regional economic stabilization of output and employment.

II THEORETICAL APPROACH – CASE OF CONSUMER SMOOTHING

In economy of complete capital markets, all risk-sharing would be realized through capital markets. Consumers would insure themselves against region-specific shocks by holding portfolios that pay higher returns when their incomes from economic activities in their own region are low. The result is that consumption would be highly correlated across regions, and interregional consumption correlations would be stronger than interregional income correlations. When capital markets are incomplete and when the representative consumers are risky averse, consumption smoothing can be provided by fiscal transfers of income across regions. That means that a central government using fiscal policy can improve the consumption in the weak region and so can shift income risk across the region. The question is the relation between intranaional risk sharing and tax rates which minimizing the variance of consumption in regions (Hagen 1998).

Proposals of the model:

- country composed $i = 1,2,...,n$ regions,
- representative consumers in each region receive income y_{it}, which is random variable with expectation y_{i0} and a fixed variance,
- in absence of the fiscal policy of the central government, the budget of consumer in each region is equal $c_{it} = y_{it}$,
- central government can use three types of fiscal instruments to equilibration the income risk across the regions:
  - state-independent taxes τ_{0i},
  - state dependent-taxes $\tau_i(y_i)$,
  - state-dependent transfers $g_i(y_i)$.

Assume that

- marginal income tax rate τ is the same in all region, so the budget balance of the central government have to satisfy the equation

$$\sum_{i=1}^{n}\left[\tau(y_{it} - y_{i0}) - g_{it}\right] = 0 = \sum_{i=1}^{n}\tau_{0i} \tag{1}$$

- transfers paid to all regions are the same $g_{it} = \tau(y_t - y_0)$, where y_t denotes average national income and y_0 is expected value.

Then the consumer's budget constrains is following:

$$c_{it} = (1-\tau)(y_{it} - y_{i0}) + \tau(y_t - y_0) - \tau_{0i} \tag{2}.$$

Under condition that exist purely on the state-dependent taxes and transfers, so fiscal policy has no instrument for risks sharing. Then is possible the optimal intranational risk sharing achieve by help of calculation tax τ which minimizing the variance of consumption c_{it}. For the i-th region has been by Jürgen [2] defined optimal tax in the form

$$\tau_i = \frac{w_i(w_i - \rho_i)}{1 + w_i(w_i - 2\rho_i)} , \tag{3}$$

where

$$w_i = \sqrt{\frac{\mathrm{var}(y_{it})}{\mathrm{var}(y_t)}} , \tag{4}$$

is the index of variance and ρ_i is the correlation between i-th region's income and the country's average income

$$\rho_{y_{it}y_t} = \frac{\mathrm{cov}\; y_{it} y_t}{\sqrt{\mathrm{var} y_{it}}\sqrt{\mathrm{var}\, y_t}} . \tag{5}$$

On the basis of equation (3) it is possible to consider: The optimal tax rate in *i*-th region depends on the correlation of its income with the countries' average income, and on the index of relative variance of its income composed to average income. If all shocks are uncorrelated and identically distributed, optimal value of intranational insurance that would ensure settlement of all stochastic income should be defined by equation

$$c_{it} = y_{i0} + (y_t - y_0). \qquad (6)$$

More generally, this is not true.

- For the regions with relation

$$w_i > 1, \qquad (7)$$

the optimal tax rate should increases with the correlation on its income with average countries income.

- If is fulfilled the form

$$\rho < \frac{2w_i}{(1 + w_i^2)}, \qquad (8)$$

the optimal tax rate increases as the variance ratio increases, i.e., high risk regions desire more insurance. In general, regions with different risk characteristics desire different tax rates.
The identical tax rate optimal for all regions is not possible to determine. The central government can on the basis of the state independent taxes implement special payments to the regions with the goal to compensate the disparity in the living standard of the inhabitant.

III Empirical Evidence

The Slovak economy is splitted on the basis of NUTS 3 Classification to eight regions. The analysis was performed over years 1997-2009. The data base is published in Regional Statistical Yearbook of Slovakia, Statistical Office of Slovakia.
Before the analysis of the relationship (3) in the Table 1 and 2 are stated the trends of the basis economic indicators for the Slovak economy and its regions in the period 1997-2009. The calculation was realized on the basis of data converted to Eur. The yearly average wage funds for whole Slovak economy and for regions were calculated on the basis of the monthly average salaries in Eur and the numbers of the employed economically active population.
The indicators in the Table 1 present basically the economic power of the regions in regards to GDP and WF. The strongest region is the Bratislava region. The weakest region is the Prešov region; even the values of indicators in the five cases out of seven are smaller. It is interesting to compare the fourth and fifth indicator; the average GDP per capita and per employed.

Table 1 Selected average indicators in the period 1997-2009; Quelle: Regional Statistical Yearbooks and own calculations

Slovak Republic	Region							
	Bratislava	Trnava	Trenčín	Nitra	Žilina	Banská Bystrica	Prešov	Košice
1. Average Gross Domestic Product (GDP) (in mil. EUR)								
39 680	10 048	4 471	4 086	4 458	4 228	3 921	3 494	4 974
2. Average Wages Fund (WF) (in mil. EUR)								
14 417	2 707	1 571	1 574	1 644	1 608	1 435	1 594	1 797
3. Share Average GDP produced per EUR Average WF (ratio)								
2,75	3,71	2,85	2,59	2,79	2,53	2,73	2,19	2,77
4. Average GDP on the one employed from the economically active population (in Eur)								
19 111	34 383	19 083	16 422	16 809	15 947	16 800	12 633	19 186
5. Average GDP per inhabitant (in Eur)								
7 358	16 509	8 071	6 773	6 275	6 093	5 952	4 401	6 464
6. Share the Average Regional GDP on the Average GDP in Slovak Economy (in %)								
-	25,32	11,27	10,3	11,23	10,65	9,88	8,81	12,54
7. Share the Regional Average WF on the Average WF in Slovak Economy (in %)								
-	18,78	10,89	10,92	11,4	11,57	9,95	11,06	12,47

Table 2 Some qualitative indicators; Quelle: Regional Statistical Yearbooks and own calculations

	average wages Eur/person/month			average growth of wages	Average unemployment rate	average net money income	average net money expenditure
	year 1997	year 2009	1997-2009	in % period 1997-2009	in % period 1997-2009	Eur/person/month period 1997-2009	
Slovenská Ekonomika	310	803	534	8,3	14,96	260	252
Bratislava	404	1104	713	8,87	6,18	340	316
Trnava	305	762	517	8,00	11,72	257	243
Trencin	291	712	487	7,77	9,11	251	244
Nitra	281	704	476	7,98	16,92	255	249
Zilina	265	798	484	8,11	13,81	249	247
Banská Bystrica	288	686	472	7,53	20,95	251	245
Presov	266	660	442	7,92	18,6	235	224
Kosice	322	772	532	7,6	20,75	255	244

In the Table 2 are presented the indicators evaluating basically the living standard of inhabitants in Slovak regions. The values of the net money incomes and net money expenditures were obtained from the regional statistical yearbooks and were recalculated. In the analyzing period, average net incomes are all higher than the average net expenditures. That means, that the inhabitants of Slovak regions did not spend more as is their net income; they did not overshoot the achieved net income. It is also interesting, that the Statistical Office of the Slovak Republic published in year 2009 that the minimal wage was 295,50 Eur (none of the regions except Bratislava region has reached it). and average wage was 774,00 Eur. Introduced average wage was reached only in regions Bratislava, Žilina and Košice.

Analysis of regions concerning the income-tax relations respectively GDP-taxes are shown in Table 3. The biggest problem was to determine the expected value of income or GDP respectively within a region and whole economy. Of course, other principle of determination of the expected value brings the different results of the analysis. In both versions of the calculation, we used the expected value as the average value for the period 1997-2009 of the annual

1st income (wage fund - total annual income of all working people of working age in the region calculated on the base of monthly salary) in mil. Eur,

2nd gross domestic product in mil. Eur.

Based on the of calculations results it can be stated (see Table 3):

1. The values of indices of variance which describe the share of the variance of regional wage funds to the variance of total wage fund (IV-WF) and the share of the regional GDP in the total GDP (IV-GDP) are maximal in the Bratislava region. The comparison of the IV-WF and IV-GDP in other regions is interesting too. In some regions is IV-WF higher than IV-GDP, what can be considered as disproportion between produced GDP and wages paid in the region. It is possible to ascertain, that in these regions the labor productivity is smaller as in other regions. This problem is connected with high unemployment rate.
2. Bratislava region over the period 1997-2009 strongly exceeds other regions. The average wage in the analyzing period was 714 Eur (534 Eur is a nation-wide average wage), the lowest unemployment rate was 6,17 % (14,96 % is a nation-wide average rate of employment), in Bratislava region one person produces 31 738 Eur GDP per year (it is app. 25 % from the average GDP in economy of Slovakia), which is almost double compared to the region of Trnava, etc. Probably a combination of economic factors such the average monthly wage, i.e. income per worker (also consumption) and the unemployment rate are substantial in the calculation of income-tax. Therefore the rate of taxation in Bratislava region is the highest of all regions. The lowest income-tax rate in the analyzing period we obtained for the region of Banská Bystrica, where is the highest unemployment rate (20,95%) and is the second lowest average monthly wage in the Slovak economy (472 Eur).

Table 3 Wage bill – taxes, GDP – taxes in the period 1997-2009; Quelle: own calculations

Slovak Republic	Region							
	Bratislava	Trnava	Trencin	Nitra	Zilina	Banska Bystrica	Presov	Kosice
Calculation on the basis of wage fund								
Average annual wage fund in mil. Eur								
1201	226	131	131	137	139	120	133	150
Index of variance in regions								
-	18,69	11,53	10,11	11,87	10,97	8,57	11,03	11,67
Correlation between region i ‘s wage fund and the country‘s wage fund								
-	0,9976	0,9968	0,9964	0,9953	0,9985	0,9984	0,9960	0,9969
Tax rate in regions								
-	22,89	12,97	11,20	13,37	12,29	9,35	12.33	13,16
Calculation on the basis of Gross domestic product								
Average annual Gross domestic product in mil. Eur								
64642	16284	7262	6662	7183	6874	6447	5712	8152
Index of variance in regions								
-	30,02	13,96	9,9798	10,67	11,38	6,83	5,70	10.08
Correlation between region i ‘s GDP and the country‘s average GDP								
-	0,9950	0,9895	0,9972	0,9818	0,9966	0,9707	0,9879	0,9928
Tax rate in regions								
-	42,32	15,96	10,82	11,65	12,78	7,07	7,99	11,09

3. Interesting comparison is obtained from the calculated rates of taxation on the basis of wage fund and GDP. The highest tax is calculated for the Bratislava region, on the basis of GDP - 42 %, reality is 19 % of direct taxes and 20 % of indirect taxes, but there exists another taxes and other charges and on the basis wage fund app. 23 %. For regions Trencin, Nitra, Banská Bystrica, Košice and Prešov, is the tax rate calculated on the basis of GDP lower than on the basis of wage fund. The highest difference of the calculated tax rate is in the Prešov region, where was in the analyzing period the lowest average monthly wage and the lowest average GDP.
4. Comparison of net monthly income per capita in Eur published in Regional Statistical Yearbook and net monthly income calculated according to equation (3) is shown in Table 4. In the first row are stated the values obtained from the Regional Statistical Yearbooks. In the third and fourth lines are calculated values of net monthly income on the basis of GDP. In the fifth and sixth lines are calculated values of net monthly income on the basis of wage fund. In the second and fourth lines are stated the calculated values of income on the basis of the same tax rate (42 %) for all region. For calculation of incomes stated in the third and fifth lines were used the different tax rates for each region.

Table 4 Average income per capita/month/Eur in period 1997-2009; Quelle: Regional Statistical Yearbook and own calculations

Regions							
Bratislava	Trnava	Trenčín	Nitra	Žilina	B.Bystrica	Prešov	Košice
Regional Statistical Yearbook							
339,96	257,47	251,32	255,34	249,40	251,49	235,03	255,28
Calculated on the basis of GDP - tax rate 42 %							
411,74	298,44	281,11	274,35	279,33	272,40	255,17	307,02
Calculated on the basis of of GDP – different tax rate in gerion							
411,74	434,82	434,63	420,24	422,38	438,88	407,05	473,26
Calculated on the basis of wage fund tax rate 23 %							
550,44	398,97	375,80	366,77	373,42	364,17	341,13	410,45
Calculated on the basis of wage fund – different tax rate in region.							
550,44	450,29	432,77	412,05	424,75	428,11	387,85	462,24

Finally it is possible to state, that in the analyzing period, the calculated values of average income per capita/month/ Eur stated in the second line are similar to the average income calculated on the basis of the published data in the Regional Statistical Yearbook.

IV Conclusion

Although different tax rates on individual regions have been obtained the in calculation, this principle is not acceptable within a unitary state in general. On the other site many large economies use not only direct subsidies from the state budget, but also different tax rates in particular for indirect taxes to offset the living standard in its regions. Many of analytical works were realized for the U.S. economy, Canada, but also Germany, Italy, France and Endgland. Obviously the principles of the given analysis could be used also for monitoring of fiscal policy in European Monetary Union. Of course the long-term stabilization of the European Monetary Union requires is not only monitoring of fiscal policy, but also its harmonization and the regulation of its principles.

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COMPARISON OF PRIORITIZATION METHODS IN ANALYTIC HIERARCHY PROCESS

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Abstract
A crucial problem in the analytic hierarchy/network process is how to derive priorities form pairwise comparison matrices. The most popular methods for deriving priorities are eigenvector method proposed originally by T. Saaty, logarithmic least square method and least square method. The paper deals with other alternative approaches using methodology of goal programming – one of them is based on minimization of sum of absolute or relative deviations and the other one on minimization of maximum deviation. The results of methods are compared on a set of randomly generated matrices of different sizes and consistency levels.

Keywords: *analytic hierarchy process, goal programming, optimization*

JEL Classification: C44
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1 INTRODUCTION

The analytic hierarchy process (AHP) was introduces by T. Saaty in 1977 – more detailed information about it can be found e.g. in (Saaty, 1990) – and since this year this method became one of the most popular tools for analysis of complex decision making problems. Even though the AHP was proposed many years ago it is still subject for research and applications – see e.g. (Jablonsky, 2007) and (Srdjevic, 2005). AHP organizes the decision problem as a hierarchical structure containing always several levels. The topmost level of the hierarchy defines the main goal of the decision problem and the lowest level usually describes the decision alternatives or scenarios. The levels in between can contain secondary goals, criteria and sub-criteria of the decision problem. The method itself is based on pairwise comparisons of elements on each level of hierarchy with respect to the elements of the preceding level. The comparisons are estimates of the preference between two elements of the lower level with respect to the element of the level above. They can be formed into a pairwise comparison matrix $\mathbf{A} = \{a_{ij} \mid a_{ji} = 1/a_{ij}, a_{ij}>0, i, j = 1, 2,..., k\}$, where k is the number of elements of the lower level. Saaty (1990) proposes to use a_{ij} integers in the range 1 through 9 to express preference of the decision maker, where 1 means that the i-th and the j-th element are equally important and 9 means that the i-th element is absolutely more important than the j-th element. Pairwise comparison matrices contain decision maker's preferences that are synthesized into local priorities of the elements of a particular level with respect to an element of the level above. By Saaty (1990) and later by other researches several methods for deriving priorities form pairwise comparison matrices are proposed. The aim of the paper is to compare some of them and suggest their modifications.
The paper is organized as follows. Section 2 presents a survey of most often used prioritization methods and discusses possibilities of their modifications. Section 3 contains information about computational experiments with randomly generated comparison matrices of different sizes and consistency levels. Some conclusions and directions for future research are given in the final section of the paper.

2 PRIORITIZATION METHODS IN THE AHP

Since formulation of principles of the AHP (and later ANP) several prioritization methods for deriving priorities from pairwise comparison matrices were proposed. The original Saaty's procedure computes the prioritization vector as the right eigenvector **w** belonging to the largest eigenvalue λ_{max} of the pairwise comparison matrix **A**. This *eigenvector method* consists in solving the following linear problem:

$$\mathbf{Aw} = \lambda_{max}\mathbf{w} \tag{1}$$

The eigenvector **w** must be normalized, i.e. $\sum_{i=1}^{n} w_i = 1$. Due to computational problems with solving of problem (1) some other prioritization methods were formulated by Saaty and later by other researchers. All of them are based on minimization of a metric (a deviation function) between elements of pairwise comparison matrices a_{ij} on one side and ratios of estimated priorities w_i/w_j on the other side.

Least square method (*LSM*) constructs the deviation function as the sum of squares of deviations between elements a_{ij} and ratios w_i/w_j, i.e. the optimization problem is as follows:

Minimize $$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{w_i}{w_j}\right)^2 \tag{2}$$

subject to $$\sum_{i=1}^{n} w_i = 1, \tag{3}$$

$w_i \geq 0$, $i = 1, 2, \ldots, n$.

The problem (2)-(3) is a difficult non-linear problem with non unique solutions that are hardly computable. That is why the *LSM* cannot be used for practical purposes. A modification of the *LSM* is the *logarithmic least square method* (*LLSM*) that minimizes the objective function

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\ln a_{ij} - \ln\left(\frac{w_i}{w_j}\right)\right)^2 \tag{4}$$

with respect to constraints (3). The solution of the problem (3)-(4) can be simply given as the geometric mean of the elements of each row of matrix A that is normalized to unit sum. That is why this method (originally proposed by Saaty) is often called *geometric mean method*. Solution of this problem is identical to the eigenvector problem (1) in case the matrix **A** is fully consistent and it is close to this solution when the consistency measure is on a satisfactory level. More about measures of consistency can be found e.g. in (Saaty, 1990).

Because of computational problems with the *LSM* method, its modification that minimizes the following metric was proposed - let us denote this method as *modified LSM* (*MLSM*):

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}w_j - v_i\right)^2 \tag{5}$$

The objective function 5 is not linear but it can be transformed into a system of linear equations – see e.g. (Bozoki, 2008) or (Gao et al., 2009).

Instead of minimization of the sum of squares it is possible to minimize the sum of positive and negative deviations or to minimize the maximum deviation. In both cases the deviations can be measured either as their absolute values or as relative deviations (in %). The optimization problem for minimization of the sum of relative deviations can be written as follows – let us denote this problem as *RSUM*:

Minimize $$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{d_{ij}^{-} + d_{ij}^{+}}{a_{ij}},$$

subject to $$a_{ij} + d_{ij}^{-} - d_{ij}^{+} = \frac{w_i}{w_j}, \quad i = 1,2,\ldots,n,\ j = 1,2,\ldots,n, \tag{6}$$

and constraints (3).

A solution that minimizes the maximum relative deviation can be given by solving the optimization problem – let us denote this problem as *RMAX*:

Minimize D,

subject to $a_{ij} + d_{ij}^{-} - d_{ij}^{+} = \frac{w_i}{w_j}, \quad i = 1,2,...,n, j = 1,2,...,n,$ (7)

$$\frac{d_{ij}^{-} - d_{ij}^{+}}{a_{ij}} \leq D, \qquad i = 1,2,...,n, j = 1,2,...,n,$$

and constraints (3).

Main group of constraints in both problems (6) and (7) are non-linear but their solution can be given quite simply by any non-linear solver, e.g. included in modeling and optimization system LINGO.

To avoid non-linearity in the models (6) and (7) their simplified version can be formulated and solved. The model for minimization of deviations is as follows:

Minimize $$\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}w_j - w_i\right| \tag{8}$$

subject to constraints (3). The model that minimizes maximum deviation is

Minimize $$\max_{i,j}\left|a_{ij}w_j - w_i\right| \tag{9}$$

subject to constraints (3). The models (8) and (9) are not linear but it is possible to re-formulate them using deviational variables into linear models very easily. The solution of models (6) and (7) on one side and models (8) and (9) on the other side is identical only for consistent matrices.

Formulation of all models in this section assumes that all elements of matrix **A** are taken into account either in constraints or the objective function. Due to the reciprocal nature of the pairwise comparison matrix **A** it is questionable whether to consider all elements or the elements greater or equal 1 only, i.e. $a_{ij} \geq 1$. All the models presented in this section can be modified accordingly.

3 COMPUTATIONAL EXPERIMENTS

The models for deriving priorities presented in the previous section were tested on randomly generated matrices of different sizes and different consistency levels. Due to the limited space for the paper we are going to present some results for pairwise comparison matrices of four elements only. Consistency indices (*CI*) of generated matrices are from very small values (0.01) until values that indicate inconsistent matrices (more than 0.1; the largest value was approx. 0.2). Table 1 presents priorities derived by six methods – eigenvector method, *LLSM*, minimization of the sum of absolute and relative deviations (*ASUM* and *RSUM*) and minimization of the maximum deviation (absolute *AMAX* and relative *RMAX*) – for one almost consistent matrix (*CI* approx. 0.015). All the optimization problems are solved for elements of the generated matrix greater or equal 1 only. Each method is described in Table 1 by four characteristics: sum of absolute deviations of elements original elements a_{ij} and ratios w_i/w_j (*ASUM*), sum of relative deviations (*RSUM*), maximum absolute deviation (*AMAX*) and maximum relative deviation (*RMAX*). The results show that the original eigenvector procedure has average values of all four characteristics and is not the best in any of them. Differences in priorities are quite high regarding to low *CI* value and applying other methods than the first two can lead to quite different final results in evaluation of alternatives, scenarios, etc.

Table 1 Results for an almost fully consistent matrix (CI = 0.015)

	Eig.val.	***LLSM***	***ASUM***	***RSUM***	***AMAX***	***RMAX***
w_1	0.5052	0.5047	0.4706	0.4706	0.4807	0.5068
w_2	0.3328	0.3322	0.3529	0.3529	0.3503	0.3245
w_3	0.1057	0.1061	0.1177	0.1177	0.1038	0.1039
w_4	0.0563	0.0570	0.0588	0.0588	0.0652	0.0648
ASUM	3.30	3.27	**1.67**	**1.67**	3.55	3.75
***RSUM* [%]**	84.62	85.19	**58.33**	**58.33**	101.43	104.19
AMAX	0.97	0.87	1.00	1.00	**0.63**	0.99
***RMAX* [%]**	24.08	24.02	33.33	33.33	31.39	**21.92**

Table 2 contains the same information as Table 1 but for a matrix that is inconsistent and its *CI* slightly exceeds the recommended threshold 0.1. It is clear and understandable that all optimization criteria are much worse for matrices with higher *CI*. The derived priorities are very different for all of the methods except the first two ones. It is quite surprising that eigenvector method and *LLSM* is worse in all four criteria than almost all other methods. Maximum difference in *LLSM* is nearly 7 and it is really extremely high value. This fact leads to question whether the standard methods are acceptable for inconsistent or nearly inconsistent matrices. This question is difficult to answer but all computational experiments show that the other methods are much better. On the other hand they are much computationally demanding and that is why their real applications depend on availability of appropriate powerful solvers.

Table 2 Results for an inconsistent matrix (CI = 0.107)

	Eig.val.	***LLSM***	***ASUM***	***RSUM***	***AMAX***	***RMAX***
w_1	0.5634	0.5604	0.4038	0.5526	0.4480	0.5085
w_2	0.2636	0.2721	0.4038	0.2763	0.3639	0.2832
w_3	0.1309	0.1271	0.1346	0.0921	0.1427	0.1587
w_4	0.0421	0.0404	0.0578	0.0790	0.0454	0.0496
ASUM	13.55	13.38	**8.67**	9.33	11.81	12.56
***RSUM* [%]**	239.74	236.03	177.78	**163.89**	224.55	238.84
AMAX	6.37	6.87	3.67	4.83	**2.86**	3.26
***RMAX* [%]**	90.98	98.19	66.67	80.56	58.97	**46.61**

4 CONCLUSIONS

Analytic hierarchy process (and its generalization which is analytic network process) belongs to one of the most popular methods for structuring and analysis of complex decision making problems. Both the methods are based on deriving priorities for the elements on each level of hierarchy by pairwise comparisons. The paper presents a survey of possible methods for deriving priorities. The standard methods as *eigenvector method* or *LLSM* are both computationally simple but they does not reach acceptable values of the optimization criteria as maximum deviation, sum of deviations, etc. The alternative procedures seem to be better with respect to all optimization criteria but they are computationally more demanding and their wider using is questionable for users without advanced background in optimization. Future research can be focused on analysis of rank reversals in case different prioritization methods are used or are combined in solving a decision problem.

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REGULAR POLYGON LOCATION PROBLEM WITH IP-SOLVER

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Abstract
The regular polygon location problem belongs to the family of problems connected with non-investment increase of public transport attractiveness. The problem arises, where circulation of buses or other public transport vehicles must be coordinated to reach regular distribution of vehicle arrivals at a bus stop. Due to non-linearity and discreteness of associated models, only heuristics have been used to solve this problem. In this contribution, we present an exact approach based on usage of particular characteristics of the problem and thorough model building. This approach together with new possibilities offered by used optimization environment enable us to solve some instances of the problem to optimality.

Keywords: *regular polygon, location problem, public transport, coordination of vehicle arrivals, free order of objects.*

JEL Classification: C61
AMS Classification: 90C27

1 INTRODUCTION

The problem originated in the field of public transport [2, 4, 8]. An original goal was to increase attractiveness of public transport by making schedule of urban and sub-urban transport more regular at some selected stops. It was taken into account that a regularity of vehicle arrivals optimizes a transportation supply for passengers by non-investment way [1, 3, 5, 8]. It was found that individual vehicles as buses, trams or trolleybuses circle along their lines in the associated urban transportation network and in addition an average time of traversing a cycle is relatively short. Under these circumstances, the same vehicle usually appears at an observed stop several times in a given period. All the vehicle arrivals form a transportation supply for the passengers coming at the stop. If some arrivals follow closely one after other, the second one of the arrivals does not contribute to the transportation supply. On the other hand, long intervals between arrivals cause an unpleasant time loss for passengers, which come at the stop randomly. It follows that some regularity of the arrivals is desirable. The regularity of vehicle arrivals at the given stop can be improved by a shift of arrival time of an individual vehicle. As a given vehicle appears at the stop several times in the given period, a shift of one of its arrivals causes shifts of all its arrivals in the period. Furthermore, it must be considered that the observed stop can be served by vehicles, which traverse different lines. It causes that time intervals between neighboring arrivals of different vehicles differ. If the observed period is long enough to be divisible by circle time of each considered vehicle, then arrival times of a given vehicle can be depicted as vertices of a regular polygon on a circle, whose circumference is equal to length of the period. That is for; we can call the problem as location of vertices of polygons on circle or briefly regular polygon location problem. In the problem, the goal is to locate the set of regular polygons in a circle so that all vertices lie on the same circumference and their distribution be regular in the sense [6, 10]. Many researchers tackled this problem in several recent decades but due to non-linearity and discreteness of associated models, only heuristics have been used to solve this problem. In this contribution, we present an exact approach based on usage of particular characteristics of the problem and thorough model building. This approach together with new possibilities offered by used optimization environment enable us to solve some instances of the problem to optimality [11].

2 FORMULATION OF THE REGULAR POLYGON LOCATION PROBLEM

Let us consider r regular polygons with the same radius and center. It follows that all polygon vertices lie on one circle. Let the p-th polygon have n_p vertices. In reminder of this paper, we assume that polygons are numbered in accordance to their decreasing numbers n_p. Vertex locations of the polygon p on the circle are uniquely given by an angle between a zero point on the circle and the first vertex of the polygon. Let T denote the circumference of the circle given in some angle units and let $d_p=T/n_p$ hold. If we introduce a variable x_p, which denotes the angle between the zero point and the first vertex of the p-th polygon, then the second vertex has location d_p+x_p, the third vertex has location $2d_p+x_p$ and so on. For the better clarity of the following models, we assign fixed and unique index to each involved polygon vertex. The j-th vertex of the p-th polygon obtains index k given by (1).

$$k = j + \sum_{q=1}^{p-1} n_p \qquad (1)$$

We also define a reverse mapping $p(k)$, which returns the index of polygon, which contains the vertex k. The total number of involved vertices is denoted as m. It is obvious that range $<0, d_p)$ is sufficient for x_p to cover all possible locations of the p-th polygon vertices. Now we can assign the lowest location a_k to each vertex $k=1, \ldots, m$ and state that current location of vertex k is given by $a_k+x_{p(k)}$ and this location varies over range $< a_k, a_k+d_{p(k)})$. The regular polygon location problem can be formulated as a search for such vector $< x_1, \ldots, x_r >$, which corresponds to the most regular distribution of the vertices along the period T. The first question, which must be answered, is: „How should the regularity be measured?" In the history of this and other related problems, several approaches to regularity were used. The first approach was based on a minimal difference between locations of neighboring vertices, which was to be maximized [10]. The second hierarchical approach was formulated as subsequent minimization of the maximal difference between locations of neighboring vertices, then the second maximal difference and so on. The third approach evaluates regularity as a sum of squares of the differences between neighboring vertex locations.

In spite of the non-linearity, we focus here on the third criterion. Let us introduce variable t_k to denote the vertex location difference between the k-th and directly preceding vertices. The index of the vertex preceding the vertex k is denoted as $i(k)$. Furthermore, we realize that the variable x_1 can be set to zero without loss of optimality. This way, t_1 can be defined as $T - a_{i(1)} - x_{p(i(1))}$, where $i(1)$ is an index of the vertex with the highest value of location in the period T. The other variables t_k for $k = 2, \ldots, m$ must satisfy the constraint $t_k = a_k + x_{p(k)} - a_{i(k)} - x_{p(i(k))}$. After these preliminaries a model of the regular polygon location problem can be formally formulated as:

$$\textit{Minimize} \quad \sum_{k=1}^{m} (t_k)^2 \qquad (2)$$

$$\textit{Subject to} \quad T - a_{i(1)} - x_{p(i(1))} = t_1 \qquad (3)$$

$$a_k + x_{p(k)} - a_{i(k)} - x_{p(i(k))} = t_k \quad \textit{for } k=2,..,m \qquad (4)$$

$$x_p < d_p \quad \textit{for } p=2,..,r \qquad (5)$$

$$x_1 = 0,\ x_p \geq 0 \ \textit{for } p=2,..,r \qquad (6)$$

Two obstacles involved in the model (2)-(6) prevent us from usage of any linear programming solver [9, 12]. The first of them is non-linearity of the objective function (2). Under some practical assumptions, we can comply this objective function non-linearity. We come out from the fact that most of vehicle schedules corresponding to the individual polygons are given in some integer time units. Thus, we can restrict ourselves only to integer values of x_p and t_k. The second assumption is that no two vertices may share the same location, what means that $t_k \geq 1$ for $k=1, \ldots, m$. Now, we can introduce non-negative auxiliary variables u_{kj} for $k=1, \ldots, m$, $j=2, \ldots, d_{p(k)}$ and use them in substitution (7) for t_k.

$$t_k = 1 + \sum_{j=2}^{d_{p(k)}} u_{kj} \qquad \textit{for } k=1,..,m \tag{7}$$

$$u_{kj} \le 1 \qquad \textit{for } k=1, ..., m, j=2,..,d_{p(k)} \tag{8}$$

The square of t_k can then be approximated by the expression (9).

$$(t_k)^2 = 1 + \sum_{j=2}^{d_{p(k)}} (2j-1)u_{kj} \qquad \textit{for } k=1,..,m \tag{9}$$

If the values of u_{kj} are integer, then the right-hand-side of (9) represents the exact value of $(t_k)^2$. Using substitutions (7) and (9), the first obstacle is overcome. The most serious obstacle is the non-linearity involved in usage of precedence mapping (permutation) *i(k)* used in (3) and (4). The model (2) – (6) after applying substitutions (7) and (9) would be completely linear if the precedence mapping (permutation) *i(k)* were fixed. Unfortunately the order of vertices changes in jumps, when the values of x_p vary. A generalization of the model (2) – (9) to case of free order *i(k)* is given in the next section.

3 MATHEMATICAL MODEL OF VERTEX ORDERING

To model ordering of the polygon vertices along the period *T*, we introduce auxiliary zero-one variables $w_{ik} \in \{0, 1\}$ for each relevant pair (*i, k*) *i*=1, …, *m*, *k*=1, …, *m*, $i \neq k$. A variable w_{ik} takes the value of one if and only if the vertex *i* directly precedes the vertex *k*. To describe the relevant pairs in the following model, we introduce a logical function *exists* defined on all pairs $(i, k) \in \{1, \dots, m\} \times \{1, \dots, m\}$. The function *exists(i,k)* takes the value of true, if and only if the pair (*i, k*) is relevant. Making use of the before introduced variables x_p and u_{kj}, a linear model of the regular polygon location problem can be formulated as follows.

$$\textit{Minimize} \qquad \sum_{k=1}^{m} \left(1 + \sum_{j=2}^{d_{p(k)}} (2j-1)u_{kj}\right) \tag{10}$$

$$\textit{Subject to} \qquad \sum_{\substack{i=1 \\ exists(i,k)}}^{m} w_{ik} = 1 \quad \textit{for } k=1,..,m \tag{11}$$

$$\sum_{\substack{i=1 \\ exists(k,i)}}^{m} w_{ki} = 1 \quad \textit{for } k=1,..,m \tag{12}$$

$$x_1 = 0, x_p \le d_p - 1 \qquad \textit{for } p=2,..,r \tag{13}$$

$$T - a_i - x_{p(i)} \ge 1 + \sum_{j=2}^{d_{p(1)}} u_{1j} - T^*(1 - w_{i1}) \quad \textit{for } i=2,..,m, \textit{ exists}(i,1) \tag{14}$$

$$T - a_i - x_{p(i)} \le 1 + \sum_{j=2}^{d_{p(1)}} u_{1j} + T^*(1 - w_{i1}) \quad \textit{for } i=2,..,m, \textit{ exists}(i,1) \tag{15}$$

$$a_k + x_{p(k)} - a_i - x_{p(i)} \ge 1 + \sum_{j=2}^{d_{p(k)}} u_{kj} - T^*(1 - w_{ik}) \textit{ for } k=2,..,m, i=1,..,m, \textit{ exists}(i,k) \tag{16}$$

$$a_k + x_{p(k)} - a_i - x_{p(i)} \le 1 + \sum_{j=2}^{d_{p(k)}} u_{kj} + T^*(1 - w_{ik}) \textit{ for } k=2,..,m, i=1,..,m, \textit{ exists}(i,k) \tag{17}$$

$$u_{kj} \le 1, \; u_{kj} \ge 0 \qquad \textit{for } k=1, ..., m, j=2,..,d_{p(k)} \tag{18}$$

$$x_p \in Z^+ \qquad \textit{for } p=1,..,r \tag{19}$$

$$w_{ik} \in \{0,1\} \qquad \textit{for } i=1,..,m, k=1,..,m, \textit{ exists}(i,k) \tag{20}$$

The consistency constraints (11) and (12) ensure that each vertex *k* has its predecessor and successor. The pair of associated constraints from (14) and (15) or (16) and (17) causes that if $w_{ik}=1$ for some pair (*i, k*), then the difference between the location of vertex *k* and the location of preceding vertex is equal to substituting expression (7).

4 NUMERICAL EXPERIMENTS

As the above outlined approach is based on usage of a general IP-solver and this solver performs common branch-and-bound method, it is obvious that complexity of the problem puts a tight limit on size of solved problem instances. The question arises, under which conditions can the limit be shifted behind a size of solved instance. We note to this topic that the order of vertices in the solved problem cannot be completely free due to fixed mutual positions of vertices, which belong to the same polygon. This specific relation may enable to reduce the computational complexity. In the presented numerical experiments, we focus on finding a dependence of computational time on the number of considered possibilities of precedence. The number of considered possibilities corresponds with the number of introduced variables w_{ik}. We performed experiments with series of instances, when each instance consists of at least three and at most four polygons. No polygon repeats in the series of polygons which form one instance. The polygons are ordered in descending order of number of vertices in the input instance data structure. Generated instances are divided into six groups in accordance to the total number of vertices. A group is characterized by a range (Range) in which the total numbers of instance vertices may vary. In addition the average total number of instance vertices (AvgV) is given for each group. Two sequences of the numerical experiments are reported. The first sequence called "Basic" uses full set of variables w_{ik}, where $i \neq k$. The second sequence of experiments denoted as "Reduced" was performed with such relevant variables w_{ik}, where the associated pair i, k of vertices satisfies the constraint $a_i < a_k + d_{p(k)}$.

All results are reported in the table 1 in the corresponding rows, where average computational time *AvgCT* necessary for obtaining an exact solution of the problem is given in seconds. The experiments were performed using the optimization software FICO Xpress 7.1 (64-bit, release 2010). The associated code was run on a PC equipped with the Intel Core i5 2430M processor with the parameters: 2.4 GHz and 4 GB RAM

Table 1 Average computational times in seconds and average numbers of introduced variables are reported in rows denoted as w_{ij} and *AvgCT* respectively. Each group consists of exactly twelve instances. Maximal and minimal times are given in rows *maxCT* and *minCT*.

	Group	1	2	3	3	4	5
	RangeV	14-20	21-27	28-34	35-41	42-48	49-55
	AvgV	17.2	24.3	31.0	37.8	44.9	52.2
Basic	w_{ij}	282	568	935	1392	1977	2673
	AvgCT	4	41	2852	369	5626	2505
	maxCT	14	137	30716	3607	25098	23620
	minCT	0.4	1	1	3	5	4
Reduced	w_{ij}	165	317	508	745	1046	1403
	AvgCT	6	8	682	150	1978	625
	maxCT	22	23	5050	627	6838	3491
	minCT	0.5	1	2	3	3	5

5 CONCLUSIONS

Presented results of numerical experiments confirm the assumed idea that a reduction of the number of variables w_{ij} can considerably reduce the associated computational time. This relation holds even when most of introduced variable are redundant, i.e. none of them can take the value of one. Also some tendency of computational time to growth, when the total number of instance vertices increases can be observed. Nevertheless, this tendency is disturbed by several singularities, which may originate in some hidden characteristics of the solved instances. Identification of these characteristics will be a topic of further research.

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ROUTING AND SCHEDULING TRAINS AT A PASSENGER RAILWAY STATION

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Abstract
The paper deals with the problem of routing and scheduling trains at a passenger railway station. Given the list of the trains arriving, departing, or travelling through the station, the timetable, and the detailed layout of a station, the routing problem consists in assigning a route through the station to each of the involved trains. If the arrival and departure times given by the timetable cannot be kept for safety or capacity reasons, then the times must be adjusted and the scheduling problem arises. Routing and scheduling are serious problems especially at large, busy stations with multiple platforms and multiple in-lines and out-lines. In this paper, the problem of routing and scheduling trains is formulated as a multiple criteria optimisation problem. The proposed model is validated by using the real data of Prague main station and the timetable valid for the years 2004/2005.

Keywords: *routing trains, scheduling, mixed integer programming, multiple-objective programming*

JEL Classification: C61
AMS Classification: 90C11, 90C29

1 INTRODUCTION

The problem of routing and scheduling trains at a station is a subproblem of the generation of a timetable for a railway company. The generation of a timetable is a hierarchical process. At the first stage, a preliminary timetable for the whole network is proposed. In this phase, a macroscopic viewpoint at the railway network is applied. Stations are considered as black boxes. Capacity limits of particular stations and the movement of trains inside the stations are not taken into account. Then, at the second stage, a microscopic viewpoint related to stations is applied. At every station, the network timetable is checked whether it is feasible with respect to capacity, safety and train operators' preferences. To prove the feasibility, detailed routes and schedules for the trains are generated. If desired arrival and departure times are not feasible at the microscopic level, the process returns to the first stage, where the timetable must be adjusted.

In the Czech and Slovak Republic, planning train movements through the station is done by hand, using planner's experience and a set of rules determined by a railway company. The main goal of this research is to design a more sophisticated approach which would serve as a planner's decision supporting tool and result in better routing and scheduling plans. Such an approach can play an important role especially at large, busy stations with multiple platforms and multiple in-lines and out-lines. Improvement in the plan quality results in

1. better management of train operation in the station, namely:
   a) shorter times of routes occupation by arriving and departing trains,
   b) uniform workload of the infrastructure elements, such as tracks, switches, and platforms, which leads to a more robust plan resistant to random disturbances;
2. higher service quality perceived by passengers, namely:
   a) shorter distances needed for changing trains,
   b) more appropriate platforms (platforms near to ticket sales points and to the station entrance, platforms equipped by station shops or catering etc.),

c) less probability of changing the planned platform when the train delays, which leads to a higher share of railway in public transport;
3. meeting train operators' requirements on arrival and departure times and platforms assigned to trains.

Routing and scheduling trains at a station has been studied by researchers in countries, where large, busy stations with capacity constraints can be found. Billionet addressed only the routing problem [2]. The problem was modelled using a graph theory and the integer programming formulation of the resulting graph colouring problem was solved. However, the *k* colouring problem is not indeed an optimisation problem, it means any feasible solution is acceptable and the problem formulation does not reflect the solution quality, such as route lengths or platform preferences for individual trains. In [10,11] the problem of train routing was described as a weighted node packing problem, using bivalent programming, while the solution algorithm applied the branch-and-cut method. A disadvantage of the above presented models is that the calculations connected with them are computationally too complex and time consuming. Another, practically oriented approach has given up on applying the integer programming methods, and replaced them by the heuristics, solving the scheduling and routing problems at a time [3]. The algorithm incorporates, or considers, the operational rules, costs, preferences and trade-offs, which are applied by experts creating plans manually. The shortcoming of this approach is obvious: since it is a heuristics, the optimality of the resulting plan is not guaranteed.

Other way of research, e.g. [1, 4, 8], has been directed at operational train management. In real time it is necessary to reflect the requirements of the operation burdened with irregularities, i.e. to re-schedule the arrivals and departures times, and/or re-route trains.

In this paper we propose a mixed integer programming (MIP), multiple criteria model of the routing and scheduling problem. The problem can be solved by a lexicographic approach, where particular criteria are ranked according to their importance.

2 PROBLEM FORMULATION

The problem of routing and scheduling trains, as described in the previous section, consists of the following partial issues subject to decision-making process. For each train,

- a platform track must be specified at which the train should arrive; the platform track assignment determines the route, on which the train approaches from an in-line (or from a depot) to the platform, or departs from the platform to an out-line (or to a depot),
- arrival time at the platform and departure time from the platform need to be determined.

The solution should minimise deviations from the planned arrival and departure times and maximise the total preferences for platforms and routes.

The inputs to the mathematical programming model are as follow:

1. track layout of the station, which is necessary for determining feasible platform tracks for a train and conflicting routes,
2. list of trains, where the data required for each train include:
   a) planned time of its arrival at the platform,
   b) planned time of its departure from the platform,
   c) line on which the train arrives (in-line) and departs (out-line),
   d) list of feasible platform tracks with their desirability for the train,
   e) category of the tain.

All time data are given in minutes.

Further on we present the formulation of the MIP model. First we need to explain the symbols used:

Subscripts which in the mathematical model represent objects

i, i', j train

k, k' platform track

Input parameters (constants)

t_i^{Pa} planned arrival time of train i at the platform

t_i^{Pd} planned departure time of train i

I_i in-line for train i

O_i out-line for train i

c_i category of train i; $c_i = 1$ for regional stopping trains and increases with the speed and distance travelled by the train

t^{min} minimum dwell time of a train at the platform

t^{max} maximum time interval, in which two train movements are tested for a conflict

p_{ik} preference coefficient; it reflects the desirability of the assignment of platform track k to train i

$a(l,k,l',k')$ coefficient, which has value *true*, if the route connecting line l to platform track k conflicts with the route connecting line l' to platform track k'; if there exists any route connecting line l to track k and any route connecting line l' to track k' such that these two routes do not conflict, then $a(l,k,l',k') = false$. $a(l,k,l',k') = true$ for $k = k'$ or $l = l'$. The existence of route conflicts can be identified in advance from a detailed map of the track layout.

We adopted the concept of conflicting routes and conflict solving from the source [3]. If two trains are on conflicting routes we must ensure that there is at least a required minimum headway (time interval) between them, for safety and signalling reasons. The minimum headway depends on the order, types, and lengths of the trains, on whether the trains are arriving or departing from the station, and on the platform track and line used by each train. For example, let $h(i,k,i',k')^{da}$ is the minimum headway required between train i departing from track k and the next train i' arriving at track k'. The superscripts d and a denote departure and arrival, and the order of the superscripts indicates the order of the trains, i.e., train i is followed by i'. Similarly we have $h(i,k,i',k')^{aa}$, $h(i,k,i',k')^{ad}$ and $h(i,k,i',k')^{dd}$ for combinations arrival – arrival, arrival – departure, departure – departure. We need not introduce subscripts to denote the in-lines or out-lines used by trains since for an arriving train i the in-line is already specified by I_i, and for a departing train i the out-line is specified by O_i.

The preference coefficient p_{ik} may reflect:

- operator's preferences of platforms,
- the distance of the track k to the connecting trains,
- the length of the route used by train i arriving to or departing from platform track k. The smoother and shorter the route is, the less the possibility of a conflict with other trains is, hence the probability of delay propagation decreases.

Sets of objects

K set of all platform tracks

$K(i)$ set of feasible platform tracks for train i

U set of all arriving, departing, and transit trains

$V^{aa} = \{(i,j): i,j \in U, i < j, |t_i^{Pa} - t_j^{Pa}| \leq t^{\max}\}$ set of ordered pairs of those trains that may arrive concurrently; similarly:

$V^{ad} = \{(i,j): i,j \in U, i < j, |t_i^{Pa} - t_j^{Pd}| \leq t^{\max}\}$

$V^{da} = \{(i,j): i,j \in U, i < j, |t_i^{Pd} - t_j^{Pa}| \leq t^{\max}\}$

$V^{dd} = \{(i, j) : i, j \in U, i < j, |t_i^{Pd} - t_j^{Pd}| \le t^{\max}\}$

Decision and auxiliary variables of the model

$$\text{for } i \in U, k \in K(i): x_{ik} = \begin{cases} 1 & \text{if track } k \text{ is assigned to train } i \\ 0 & \text{otherwise} \end{cases}$$

u_i real arrival time of train i at a platform, $i \in U$

v_i real departure time of train i from a platform, $i \in U$

The following auxiliary variables $\boldsymbol{y}$ are introduced for the couple of those trains i and j that may travel concurrently. They enable to express safety headways between conflicting trains.

$$\text{for } (i, j) \in V^{aa} : y_{ij}^{aa} = \begin{cases} 1 & \text{if train } i \text{ arrives before train } j \text{ arrives} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for } (i, j) \in V^{ad} : y_{ij}^{ad} = \begin{cases} 1 & \text{if train } i \text{ arrives before train } j \text{ departs} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for } (i, j) \in V^{da} : y_{ij}^{da} = \begin{cases} 1 & \text{if train } i \text{ departs before train } j \text{ arrives} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for } (i, j) \in V^{dd} : y_{ij}^{dd} = \begin{cases} 1 & \text{if train } i \text{ departs before train } j \text{ departs} \\ 0 & \text{otherwise} \end{cases}$$

Model formulation

minimise $$\sum_{i \in U} c_i (u_i - t_i^{Pa}) + \sum_{i \in U} c_i (v_i - t_i^{Pd}) \quad (1)$$

maximise $$\sum_{i \in U} \sum_{k \in K(i)} p_{ik} x_{ik} \quad (2)$$

subject to

$$u_i \ge t_i^{Pa} \qquad \forall i \in U \quad (3)$$

$$v_i \ge t_i^{Pd} \qquad \forall i \in U \quad (4)$$

$$v_i \ge u_i + t^{\min} \qquad \forall i \in U \quad (5)$$

$$u_{i'} \ge u_i + h(i,k,i',k')^{aa} - M(1 - y_{ii'}^{aa}) - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{aa}, k \in K(i), k' \in K(i') : a(I_i, k, I_{i'}, k') \quad (6)$$

$$u_i \ge u_{i'} + h(i',k',i,k)^{aa} - My_{ii'}^{aa} - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{aa}, k \in K(i), k' \in K(i') : a(I_i, k, I_{i'}, k') \quad (7)$$

$$v_{i'} \ge u_i + h(i,k,i',k')^{ad} - M(1 - y_{ii'}^{ad}) - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{ad}, k \in K(i), k' \in K(i') : a(I_i, k, O_{i'}, k') \quad (8)$$

$$u_i \ge v_{i'} + h(i',k',i,k)^{da} - My_{ii'}^{ad} - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{ad}, k \in K(i), k' \in K(i') : a(I_i, k, O_{i'}, k') \quad (9)$$

$$u_{i'} \ge v_i + h(i,k,i',k')^{da} - M(1 - y_{ii'}^{da}) - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{da}, k \in K(i), k' \in K(i') : a(O_i, k, I_{i'}, k') \quad (10)$$

$$v_i \ge u_{i'} + h(i',k',i,k)^{ad} - My_{ii'}^{da} - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{da}, k \in K(i), k' \in K(i') : a(O_i, k, I_{i'}, k') \quad (11)$$

$$v_{i'} \ge v_i + h(i,k,i',k')^{dd} - M(1 - y_{ii'}^{dd}) - M(1 - x_{ik}) - M(1 - x_{i'k'})$$

$$\forall(i,i')\in V^{dd}, k\in K(i), k'\in K(i'): a(O_i,k,O_{i'},k') \quad (12)$$

$$v_i \geq v_{i'} + h(i',k',i,k)^{dd} - My_{ii'}^{dd} - M(1-x_{ik}) - M(1-x_{i'k'})$$

$$\forall(i,i')\in V^{dd}, k\in K(i), k'\in K(i'): a(O_i,k,O_{i'},k') \quad (13)$$

$$y_{ij}^{aa} = 1 \qquad \forall i,j\in U, i\neq j, I_i = I_j, t_i^{Pa} \leq t_j^{Pa} \quad (14)$$

$$\sum_{k\in K(i)} x_{ik} = 1 \qquad \forall i\in U \quad (15)$$

$$u_i, v_i \geq 0 \qquad \forall i\in U \quad (16)$$

$$x_{ik} \in \{0,1\} \qquad \forall i\in U\ \forall k\in K(i) \quad (17)$$

$$y_{ij}^{aa} \in \{0,1\} \qquad \forall(i,j)\in V^{aa} \quad (18)$$

$$y_{ij}^{ad} \in \{0,1\} \qquad \forall(i,j)\in V^{ad} \quad (19)$$

$$y_{ij}^{da} \in \{0,1\} \qquad \forall(i,j)\in V^{da} \quad (20)$$

$$y_{ij}^{dd} \in \{0,1\} \qquad \forall(i,j)\in V^{dd} \quad (21)$$

Model description

Objective function (1) minimises the weighted deviations of the arrival and departure times proposed by the model from the times specified by the timetable. The weights cause that long-distance/high- speed trains will respect planned times and regional trains will be postponed if necessary. The second criterion maximises the desirability of the platform tracks to be assigned to the trains.

Constraints (3) – (5) put the lower bounds on the arrival and departure times. The constraints reflect the rule used in manual planning saying that if the desired timetable cannot be kept, the arrival and/or departure times of the train are increased. A minimum dwell time needed for boarding and alighting must be kept.

Constraints (6) – (13) ensure that a minimum headway will be kept between conflicting trains. More precisely, constraint (6) states that if trains i and i' have planned arrival times within t^{max} and train i arrives at platform track k before train i' arrives at track k', i.e.

$$x_{ik} = 1, x_{i'k'} = 1, y_{ii'}^{aa} = 1, \quad (22)$$

and trains are on conflicting routes (i.e. $a(I_i,k,I_{i'},k')$ is *true*), then train i' is allowed to arrive at least $h(I_i,k,I_{i'},k')^{aa}$ minutes later than train i. If at least one of the conditions (22) is not met (e.g. train i is not assigned to track k), then constraint (6) becomes irrelevant as the right-hand side is negative (M is a suitably picked high positive number). If train i' is followed by train i ($y_{ii'}^{aa} = 0$), then i is allowed to arrive at least $h(I_{i'},k',I_i,k)^{aa}$ minutes later than i', which is ensured by constraint (7). Constraints (8) – (13) have a similar meaning for the other combinations of arrival – departure.

Constraint (14) states that y_{ij}^{aa} is 1 if train i is followed by train j at the arrival and both trains travel on the same in-line.

Constraint (15) ensures that each train is always dispatched to exactly one platform track.

The remaining obligatory constraints (16) – (21) specify the definition domains of the variables.

To solve this multiple-criteria optimisation problem, several methods can be used [5]. We chose the lexicographic approach, where the objective functions are ranked according to their importance. In the problem at hand, the first objective function (i.e. to meet the timetable) is more important that the second one (i.e. to respect track preferences). This ordering reflects how decisions are currently made in practice. The solution technique consists of two steps. In the first step the problem (1), (3) – (21) is solved giving the best value of the weighted sum of deviations f_1^{best}. Then the constraint

$$\sum_{i\in U} c_i\left(u_i - t_i^{Pa}\right) + \sum_{i\in U} c_i\left(v_i - t_i^{Pd}\right) \le f_1^{best} \tag{23}$$

is added and the model (2) – (21), (23) is solved. Because both MIP problems are hard and the optimal solutions cannot be found within a reasonable time limit, we decided to implement Local Branching metaheuristic [7] using the general optimisation software *Xpress* [6].

3 CASE STUDY AND EXPERIMENTS

The model was verified by using the real data of Prague main station and the timetable valid for the years 2004/2005. Prague main station is a large station that at the given time had 7 platforms, 17 platform tracks and 5 arrival/departure line tracks. According to the timetable 2004/2005, the station dealt with 376 regular passenger trains per day.

In preliminary experiments, only trains arriving or departing in the morning peak time – from 5:00 to 10:00 input the model. Using input data for 90 trains in this time period, the model has 180 continuous variables, 1938 bivalent variables, and 59745 constraints. Coefficient p_{ik} encodes the distance of track k from the track regularly assigned to train i. It has the value of 1 for the regular track and a lower value decreasing with the distance for other feasible tracks. All safety headways $h(.)^{aa}$, $h(.)^{ad}$, $h(.)^{da}$, and $h(.)^{dd}$ are set to 2 minutes.

The experiments showed that the timetable 2004/2005 was not correct with regard to safety requirements. There were some trains travelling on conflicting routes concurrently. That is why their desired arriving or departing times could not be kept. The best solution found in the total time limit of 4 hours delays 2 trains at arrival by 4 minutes and 3 trains at departure by 4 minutes, and dispatches 14 (out of 90) trains to platform tracks different from the planned ones.

A more detailed discussion of computational experiments will be presented at the conference.

4 CONCLUSIONS

In the paper, a mixed integer programming model for routing and scheduling trains at a passenger railway station is described. The model gives a solution with regard to particular criteria ranked according to their importance, i.e. the solution with minimal deviation of the arrival and departure times from the timetable that respects the desirability of the platform tracks to be assigned to the trains as much as possible. If we wanted the plan to be more robust and the probability of potential delay propagation to decrease, the platform tracks should be utilized uniformly. To achieve this requirement, another criterion reflecting track utilization irregularity would be needed. Possible irregularity measures are reported in [9].

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RISK MEASURES VIA HEAVY TAILS

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Abstract
Economic and financial activities are often influenced simultaneously by a decision parameter and a random factor. Since mostly it is necessary to determine the decision parameter without knowledge of a random element realization, deterministic optimization problems depending on a probability measure correspond often to such situations. In applications very often the problem has to be solved on the data basis. Great effort has been paid to investigate properties of these (empirical) estimates; mostly under assumptions of "thin" tails and a linear dependence on the probability measure. The aim of this contribution is to focus on the cases when these assumptions are not fulfilled. This happens usually in economic and financial applications (see e.g. [10], [12], [14],[18]).

Keywords: *Static stochastic optimization problems, linear and nonlinear dependence, thin and heavy tails.*

JEL Classification: C44
AMS Classification: 90C15

1 INTRODUCTION

Let (Ω, S, P) be a probability space; $\xi (:= \xi(\omega) = [\xi_1(\omega), \ldots, \xi_s(\omega)])$ an s–dimensional random vector defined on (Ω, S, P); $F (:= F(z), z \in R^s)$ the distribution function of ξ; P_F the probability measure corresponding to F. Let, moreover, $g_0 (:= g_0(x, z))$ be a function defined on $R^n \times R^s$; $X_F \subset R^n$ a nonempty set generally depending on F, $X \subset R^n$ a nonempty "deterministic" set. If E_F denotes the operator of mathematical expectation corresponding to F, then static "classical" stochastic optimization problem can be introduced in the form:
Find

$$\phi(F, X_F) = \inf\{\mathrm{E}_F g_0(x, \xi) \mid x \in X_F\}. \qquad (1)$$

The objective function in (1) depends linearly on the probability measure P_F. Recently appear problems that can be covered only by more general type of problems:
Find

$$\bar{\phi}(F, X_F) = \inf\{\mathrm{E}_F \bar{g}_0(x, \xi, \mathrm{E}_F h(x, \xi)) \mid x \in X_F\}, \qquad (2)$$

where $h := h(x, z) = (h_1(x, z), \ldots, h_{m_1}(x, z))$ is m_1–dimensional vector function defined on $R^n \times R^s$, $\bar{g}_0 (:= \bar{g}_0(x, z, y))$ is a real–valued function defined on $R^n \times R^s \times R^{m_1}$.

Let us recall and analyze some simple examples that can appear:

1. If $L := (L(x, z))$ (defined on $R^n \times R^s$) represents a loss function, then $VaR_\alpha(x) := \min_u \{P\{\omega : L(x, \xi) \le u\} \ge \alpha\}, \alpha \in (0,1)$ can be considered as a risk measure, known as "Value–at –Risk" (see e.g. [3]).
   Setting $X_F := \{x \in X : [\min_u P\{\omega : L(x, \xi) \le u\} \ge \alpha] \le u_0\}$, u_0 constant; we can obtain the problem with risk measure in constraints.
2. $CVaR_\alpha(x) = \min_{v \in R} [v + \frac{1}{1-\alpha} \mathrm{E}_F (L(x, \xi) - v)^+]$ is another risk measure known as "Conditional Value–at–Risk". Setting $\bar{g}_0(x, z, y) := CVaR_\alpha(x)$ we obtain function

not depending linearly on the probability measure. However, since according to [15]

$$\min_{x\in X} CVaR_{\alpha}(x) = \min_{(v,x)\in R^1\times X} \{v + \frac{1}{1-\alpha} \mathrm{E}_F (L(x,\xi)-v)^+\},$$

the dependence of the objective on the probability measure is already linear.

3. Employing Markowitz approach to very simple portfolio problem:

Find $\max \sum_{k=1}^{n} \xi_k x_k$ s.t. $\sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1,\ldots,n, \quad s = n,$

with x_k a fraction of the unit wealth invested in the asset k, ξ_k the return of the asset, we can introduce the Markowitz problem (see e.g. [2]):

Find

$$\phi^M(F) = \max\{\sum_{k=1}^{n} \mu_k x_k - K \sum_{k=1}^{n}\sum_{j=1}^{n} x_k c_{k,j} x_j\} \quad \text{s. t.} \quad \sum_{k=1}^{n} x_k \le 1, \qquad (4)$$

$$x_k \ge 0, \quad k = 1,\ldots,n, \quad K > 0 \text{ constant},$$

where $\mu_k = \mathrm{E}_F \xi_k, \quad c_{k,j} = \mathrm{E}_F(\xi_k - \mu_k)(\xi_j - \mu_j), \quad k, j = 1,\ldots n.$ The dependence on the probability measure in (4) is not linear. $\sum_{j=1}^{n} x_k c_{k,j} x_j$ can be considered as a risk measure that can be replaced, for example by $\mathrm{E}_F |\sum_{k=1}^{n} \xi_k x_k - \mathrm{E}_F[\sum_{k} \xi_k x_k]|$ (see [9]). The dependence on the probability measure is again nonlinear.

In applications often we have to replace the measure P_F by an empirical measure P_{F^N}. Consequently, (instead of the problems (1) and (2)) the following problems are solved:
Find

$$\phi(F^N, X_{F^N}) = \inf\{\mathrm{E}_{F^N} g_0(x,\xi) \,|\, x \in X_{F^N}\}. \qquad (5)$$

Find

$$\bar{\phi}(F^N, X_{F^N}) = \inf\{\mathrm{E}_{F^N} \bar{g}_0(x,\xi,\mathrm{E}_{F^N} h(x,\xi)) \,|\, x \in X_{F^N}\}. \qquad (6)$$

Solving (5) and (6) we obtain estimates of the optimal values and optimal solutions. Their investigation started in [20], followed by many papers (see e.g. [1], [5], [6], [7], [17]). There consistency, the convergence rate and asymptotic distribution have been studied under the assumptions of "weak" tails distributions, $X_F = X$ and linear dependence of objective function on the probability measure. The exception are e.g. papers [4], [8] and [14]. We focus on the problem (2), the case of "heavy" tails and $X_F := X$.

2 SOME DEFINITIONS AND AUXILIARY ASSERTIONS

Let $F_i, i = 1,\ldots s$ denote one–dimensional marginal distribution functions corresponding to F; $P(R^s)$ the set of Borel probability measures on $R^s, s \ge 1$;
$M_1(R^s) = \{P \in P(R^s) : \int_{R^s} \|z\|_s^1 P(dz) < \infty\}$; $\|\cdot\|_s^1$ denote L_1 norm in R^s. We introduce the assumptions:

B. 1. $P_F, P_G \in M_1(R^s)$, there exist $\varepsilon > 0$ such that

- $\bar{g}_0(x,z,y)$ is for $x \in X(\varepsilon), z \in R^s$ a Lipschitz function of $y \in Y(\varepsilon)$ with a Lipschitz constant L^y;
  $Y(\varepsilon) = \{y \in R^{m_1} : y = h(x,z) \quad \text{for some} \quad x \in X(\varepsilon), z \in R^s\}$, $\mathrm{E}_F h(x,\xi)$, $\mathrm{E}_G h(x,\xi) \in Y(\varepsilon)$,

- for every $x \in X(\varepsilon), y \in Y(\varepsilon)$ there exist finite mathematical expectations, $\mathrm{E}_F \bar{g}_0(x, \xi, \mathrm{E}_F h(x, \xi))$, $\mathrm{E}_F g_0^1(x, \xi, \mathrm{E}_G h(x, \xi))$, $\mathrm{E}_G g_0^1(x, \xi, \mathrm{E}_F h(x, \xi))$, $\mathrm{E}_G g_0^1(x, \xi, \mathrm{E}_G h(x, \xi))$,
- $h_i(x, z), i = 1, \ldots, m_1$ are for every $x \in X(\varepsilon)$ Lipschitz functions of z with the Lipschitz constants L_h^i (corresponding to L_1 norm),
- $\bar{g}_0(x, z, y)$ is for every $x \in X(\varepsilon), y \in Y(\varepsilon)$ a Lipschitz function of $z \in R^s$ with the Lipschitz constant L^z (corresponding to L_1 norm).

B. 2. $\bar{g}_0(x, z, y), h(x, z)$ are uniformly continuous functions on $X(\varepsilon) \times R^s \times Y(\varepsilon)$,

B. 3. X is a convex set and $\bar{g}_0(x, \xi, \mathrm{E}_F h(x, \xi))$ a convex function on $X(\varepsilon)$.

($X(\varepsilon), \varepsilon > 0$ denotes $\varepsilon -$ neighbourhood of X.)

Proposition 1. [8] Let $P_F, P_G \in M_1(R^s)$, the assumptions B.1 be fulfilled, then there exist $\hat{C} > 0$ such that it holds for $x \in X$

$$| \mathrm{E}_F \bar{g}_0(x, \xi, \mathrm{E}_F h(x, \xi)) - \mathrm{E}_G \bar{g}_0(x, \xi, \mathrm{E}_G h(x, \xi)) | \le \hat{C} \sum_{i=1}^{s} \int_{-\infty}^{\infty} | F_i(z_i) - G_i(z_i) | dz_i. \quad (7)$$

Proposition 1 reduces s–dimensional case to one dimensional. Of course a stochastic dependence between components of the random vector is there neglected. The idea to reduce s–dimensional case to one dimensional appeared already in [11].

3 PROBLEM ANALYSIS

To employ the Proposition 1 to empirical estimates we introduce the assumptions:

A.2. - $\{\xi^i\}_{i=1}^{\infty}$ is independent random sequence corresponding to F,

- F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}$,

A.3. $P_{F_i}, i = 1, \ldots s$ are absolutely continuous w. r. t. the Lebesguemeasure on the R^1.

Lemma 1 [19] Let $s = 1$, $P_F \in M_1(R^1)$ and A.2 be fulfilled. Then

$$P\{\omega : \int_{-\infty}^{\infty} | F(z) - F^N(z) | dz \xrightarrow[N \to \infty]{} 0\} = 1.$$

Proposition 2. [4], [8] Let $s = 1, t > 0$ and A.2, A.3 be fulfilled, $\aleph$ denotes the set of natural numbers. If there exists $\beta > 0, R := R(N) > 0$ defined on $\aleph$ such that $R(N) \xrightarrow[N \to \infty]{} \infty$ and, moreover,

$$N^{\beta} \int_{-\infty}^{-R(N)} F(z) dz \xrightarrow[N \to \infty]{}, \qquad N^{\beta} \int_{R(N)}^{\infty} [1 - F(z)] \xrightarrow[N \to \infty]{} 0,$$

$$2NF(-R(N)) \xrightarrow[N \to \infty]{} 0, \qquad 2N[1 - F(R(N))] \xrightarrow[N \to \infty]{} 0, \quad (8)$$

$$\left(\frac{12 N^{\beta} R(N)}{t} + 1\right) \exp\{-2N\left(\frac{t}{12 R(N) N^{\beta}}\right)^2\} \xrightarrow[N \to \infty]{} 0,$$

$$\text{then} \quad P\{\omega : N^{\beta} \int_{-\infty}^{\infty} | F(z) - F^N(z) | dz > t\} \xrightarrow[N \to \infty]{} 0. \quad (9)$$

Evidently, the validity of the relation (9) depends on the tails behaviour.

Proposition 3. [4] Let $s = 1, t > 0, r > 0$ and A.2, A.3 be fulfilled. Let moreover ξ be a random variable such that $\mathrm{E}_F |\xi|^r < \infty$. If constants $\beta, \gamma > 0$ fulfil the inequalities $0 < \beta + \gamma < 1/2, \quad \gamma > 1/r, \quad \beta + (1-r)\gamma < 0$, then the relation (9) is valid.

4 MAIN RESULTS

Applying the assertions of former parts we obtain.

Theorem 1. Let the assumptions B.1, A.2, A.3 and either B.2 or B.3 be fulfilled, X be a compact set and $P_F \in M_1(R^s)$. Then

$$P\{\omega : |\bar{\phi}(F^N, X) - \bar{\phi}(F, X)| \xrightarrow[N\to\infty]{} 0\} = 1.$$

Proof. The assertion of Theorem 1 follows from Proposition 1 and Lemma 1.

If $f_i, i = 1, \ldots, s$ denote the probability densities corresponding to F_i, then it holds.

Theorem 2. Let the assumptions B.1, A.2, A.3 be fulfilled, $P_F \in M_1(R^s), t > 0$. If

- for some $r > 2$ it holds that $\mathrm{E}_{F_i} |\xi_i|^r < +\infty, \quad i = 1, \ldots, s$,
- $\beta, \gamma > 0$ fulfil the inequalities $0 < \beta + \gamma < 1/2, \quad \gamma > 1/r, \quad \beta + (1-r)\gamma < 0$,

then

$$P\{\sup_{x\in X} N^{\beta} |\mathrm{E}_{F^N} \bar{g}_0(x, \xi, \mathrm{E}_{F^N} h(x,\xi)) - \mathrm{E}_F \bar{g}_0(x, \xi, \mathrm{E}_F h(x,\xi))| > t\} \xrightarrow[N\to\infty]{} 0. \tag{10}$$

If moreover either B.2 or B.3 is valid and X is a compact set , then also

$$P\{\omega : N^{\beta} |\bar{\phi}(F, X) - \bar{\phi}(F^N, X)| > t\} \xrightarrow[N\to\infty]{} 0. \tag{11}$$

Proof The first assertion follows from Propositions 1,2,3. The second assertion follows from first one and from the properties of the convex functions and the integrals. (See a similar proof for the problem (1) in [4]).

Evidently, the convergence rate $\beta := \beta(r)$ introduced by Theorem 2 depends on the absolute moments existence; it holds that $\beta(r) \xrightarrow[r\to\infty]{} 1/2, \quad \beta(r) \xrightarrow[r\to 2^+]{} 0$. Consequently, the best convergence rate is valid not only for exponential tails but also for every distribution with finite all absolute moments (e.g. Weibull and lognormal); even in the case when finite moment generating function does not exist. Unfortunately we can not obtain (by this approach) any results in the case when there exist only finite $\mathrm{E}_F |\xi_i|^r, i = 1, \ldots, s$ for $r < 2$. This is the case of stable distributions (with exception of normal distribution) or the case of Pareto distribution with a shape parameter $\alpha \le 2$.

5 CONCLUSION

The paper generalizes the results concerning the rate convergence of empirical estimates of static stochastic optimization problems depending linearly on probability measure [4] to the case when this assumption is not fulfilled. The corresponding simulation results presented in [4] can be employed in this more generalized case also. Employing some growth conditions (see e.g. [16]) the introduced results can be transformed to the estimates of the optimal solution. However the investigation in this direction as the investigation in the case of stable distributions is beyond the scope of this paper.

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EFFICIENCY OF SEVERAL RISK MINIMIZING PORTFOLIOS

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Abstract
Portfolio selection problem and its efficiency evaluation is one of the most important issues within financial risk management and decision making. Therefore, the alternative ways of portfolio comparisons were developed, among them the second order stochastic dominance (SSD) approach is one of the most popular one. The task of this paper is to examine and analyze the SSD efficiency of min-var portfolios that are selected on the basis of alternative concordance matrices set up on the basis of either Spearman rho or Kendall tau. It is empirically documented that only Pearson measure in Markowitz model identified a portfolio that can be of interest for at least one risk averse investor. Moreover, a portfolio based on Kendall measure is very poor (at least in terms of SSD efficiency).

Keywords: *stock index, concordance measure, stochastic dominance*
JEL Classification: G11, C44
AMS Classification: 90B50, 90C15

1 INTRODUCTION

The portfolio selection problem is one of the most important issues of financial risk management. In order to determine the optimal composition for a particular portfolio it is crucial to estimate the dependency among the evolution of particular risk factors, i.e., the joint distribution of log-returns of particular assets. However, in order to formulate the joint distribution, there is a need for a suitable measure of dependency. A standard assumption is that the (joint) distribution of large portfolios is multivariate normal and that the dependency can be described well by a linear measure of correlation (Pearson coefficient of correlation). Unfortunately, from real applications it is clear that the Pearson correlation is not sufficiently robust to describe the dependency of market returns (see e.g. [18]).

Among more advanced candidates for a suitable dependency measure we can classify the well-known concordance measures such as Kendall’s tau or Spearman’s rho. Minimizing these alternative measures of portfolio’s risk one can obtain several distinct “optimal” portfolios. The question is how to compare these portfolios among each other.

Efficiency with respect to stochastic dominance offers an attractive portfolio evaluation approach. Stochastic dominance was introduced in e.g. [4]. The definition of second-order stochastic dominance (SSD) relation uses comparisons of either twice cumulative distribution functions, or expected utilities (see for example [10]).

Similarly to the well-known mean-variance criterion, the second-order stochastic dominance relation can be used in portfolio efficiency analysis as well. A given portfolio is called SSD efficient if there exists no other portfolio preferred by all risk-averse and risk-neutral decision makers (see for example [9] or [6]). To test SSD efficiency of a given portfolio relative to all portfolios created from a set of assets [17], [9] and [6] proposed several linear programming algorithms.

In this paper we try to examine the efficiency of selected portfolios by terms of SSD because we assume risk averse decision makers. Our main idea is that there might by some impact of (*i*) alternative dependency measures and/or (*ii*) shortselling constrains on the efficiency of a min-var portfolio. Therefore, we identify several distinct min-var portfolios on the basis of alternative concordance matrix as defined in [16]. We also consider two types of restrictions on short sales (Black model and Markowitz model) and three measures of dependency/concordance (Pearson, Spearman and Kendall), so that we get 6 distinct portfolios in total. We apply the Kuosmanen

SSD efficiency test to these portfolios in order to analyze their SSD efficiency. More particularly, the SSD efficiency/inefficiency measure is evaluated for each portfolio and the impact of short sales restriction and choice of measure of concordance on the SSD efficiency/inefficiency of min-var portfolios is analyzed.

We proceed as follows. First, in Section 2, we summarize the basic theoretical concepts of concordance measures and portfolio selection problem. Next, in Section 3, stochastic dominance approach with a special focus on portfolio efficiency with respect to SSD criterion is presented. In Section 4, we continue with a numerical study: We identify 6 min-var portfolios first and then we test their SSD efficiency. Finally, we calculate SSD efficiency/inefficiency measures of these portfolios to be able to compare their SSD performance. In Section 5 the most important conclusions and remarks are stated.

2 CONCORDANCE MEASURES AND PORTFOLIO SELECTION

Let us consider a random vector $\mathbf{r} = (r_1, r_2, \dots, r_n)'$ of returns of n assets with discrete probability distribution described by T equiprobable scenarios. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \dots, x_n^t)$ is the t-th row of matrix X. We will use $\mathbf{w} = (w_1, w_2, \dots, w_n)'$ for the vector of portfolio weights. Throughout the paper, we will consider two special sets of portfolio weights:

$$W_M = \left\{ \mathbf{w} \in R^n : \sum_{i=1}^{n} w_i = 1, w_i \geq 0, i = 1, 2, \dots, n \right\}$$

$$W_B = \left\{ \mathbf{w} \in R^n : \sum_{i=1}^{n} w_i = 1, w_i \geq -1, i = 1, 2, \dots, n \right\}.$$

Besides that, we use the following notation: expected returns $\mathbf{m} = (\mu_1, \mu_2, \dots, \mu_n)'$, standard deviations of returns $\mathbf{s} = (\sigma_1, \sigma_2, \dots, \sigma_n)'$, and correlation matrix $R = [\rho_{i,j}]$, i.e. it consists of all combinations of Pearson linear coefficient of correlation ρ_{ij}, where $i, j = 1, \dots n$.

Following the standard portfolio selection problem of Markowitz [13], [14] no riskless investment is allowed and only the mean return and the risk measure of standard deviation matter, mainly since the Gaussian distribution of price returns is assumed. In such a setting, the efficient frontier of portfolios, i.e., the only combination of particular assets that should be considered for risky investments is bounded by minimal variance portfolio, Π_A, from the left and maximal return portfolio, Π_B, from the right. We can obtain them as follows.

Task 1 *Minimal variance portfolio,* Π_A

$$\begin{aligned} &\operatorname{var}(\Pi) \to \min, \quad \text{with} \quad && \operatorname{var}(\Pi) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \\ & && \Sigma = [\sigma_i \sigma_j \rho_{ij}] \\ & \text{s.t.} && \mathbf{w}'\mathbf{1} = 1 \\ & && \mathbf{w} \geq 0. \end{aligned}$$

Task 2 *Maximal return portfolio,* Π_B

$$\begin{aligned} &\mu(\Pi) \to \max, \quad \text{with} \quad && \mu(\Pi) = \mathbf{w}'\mathbf{m} \\ & \text{s.t.} && \mathbf{w}'1 = 1 \\ & && \mathbf{w} \geq 0. \end{aligned}$$

Alternatively, Task 1 (Task 2) can be solved subject to $w_i \geq -1$, $i = 1, 2, \dots, n$, i.e., short positions in any of the assets are allowed with no restriction on long positions (*Black model* [1]). The optimal portfolio under both models depends on preferences of a particular investor.

Obviously, the composition of any portfolio, except the maximal return one, will depend on the correlation matrix. The elements of the correlation matrix R, i.e., a crucial factor to determine optimal weights for Π_A, describe the linear dependency among two variables. The main drawback is that it can be zero even if the variables are dependent and it does not take into account tail dependency. It follows that the correlation is suitable mainly for problems with elliptically distributed random variables. Since the assumption about the Gaussianity of financial returns is unjustifiable – this observation goes back to early 60's, see e.g. [11-12] or [3] – there is a need for alternative measures, which should allow us to obtain better performance, diversification or both.

A general family of measures that is not restricted to the case of linear dependency consists of *concordance measures*. A measure of concordance is any measure that is normalized to the interval [–1,1] and pays attention not only to the dependency but also to the co-monotonicity and anti-monotonicity. For more details on all properties of concordance measures see e.g. [15].

Following [15], two random variables (X,Y) with independent replications (x_1,y_1) and (x_2,y_2) are concordant if $x_1 < x_2$ $(x_1 > x_2)$ implies $y_1 < y_2$ $(y_1 > y_2)$. Similarly, the two variables are discordant if $x_1 > x_2$ $(x_1 > x_2)$ implies $y_1 < y_2$ $(y_1 < y_2)$. The concordance measures are easily definable by copula functions, since they rely only on the "joint" features, having no relations to the marginal characteristics. There are two popular measures of concordance – Kendall's tau and Spearman's rho, which can be accompanied by the following measures of association: Gini's gamma and Blomqvist's beta. The first measure of concordance in mind is Kendall's tau, τ_K, since it is defined as the probability of concordance reduced by the probability of discordance:

$$\tau_K(X,Y) = \mathrm{P}\big((x_1 - x_2)(y_1 - y_2) > 0\big) - \mathrm{P}\big((x_1 - x_2)(y_1 - y_2) < 0\big), \quad (1)$$

with the following simplification for continuous variables:

$$\tau_K(X,Y) = 2P\big((x_1 - x_2)(y_1 - y_2) > 0\big) - 1. \quad (2)$$

For n observations it can be estimated on the basis of observations of concordance (c) and disconcordance (d) as follows:

$$\tau_K(X,Y) = \frac{c-d}{c+d}. \quad (3)$$

In order to define the second popular measure of concordance, Spearman's rho, ρ_S, the third realization of both random variables, (x_3,y_3), should be considered:

$$\rho_S = 3\mathrm{P}\big((x_1 - x_2)(y_1 - y_3) > 0\big) - \mathrm{P}\big((x_1 - x_2)(y_1 - y_3) < 0\big), \quad (4)$$

It means that the Spearman's rho is given as the probability of concordance reduced by the probability of discordance, in contrast to the Kendall's tau, for the pairs (x_1,y_1) and (x_2,y_3).

Thus, being equipped with formulas to calculate (estimate) alternative dependency measures, we can replace the elements of the covariance matrix Σ from Task 1:

$$\mathrm{cov}(X,Y) = \sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}\rho(X,Y)$$

by the elements of a concordance matrix e.g. by terms of Spearman's rho (4):

$$\mathrm{cov}_S(X,Y) = \sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}\rho_S(X,Y),$$

or Kendall's tau (1):

$$\mathrm{cov}_K(X,Y) = \sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}\rho_K(X,Y).$$

3 SECOND ORDER STOCHASTIC DOMINANCE AND PORTFOLIO EFFICIENCY

Stochastic dominance relation allows comparison of two portfolios via comparison of their random returns. Let $F_{\mathbf{r'w}}(x)$ denote the cumulative probability distribution function of returns of portfolio with weights $\mathbf{w}$. Since each portfolio is uniquely given by its weight vector we will shortly denote this portfolio by $\mathbf{w}$, too. The twice cumulative probability distribution function of returns of portfolio $\mathbf{w}$ is given by:

$$F^{(2)}_{\mathbf{r'w}}(t) = \int_{-\infty}^{t} F_{\mathbf{r'w}}(x)dx.$$

We say that portfolio **v** dominates portfolio **w** by second-order stochastic dominance ($\mathbf{r'v} \succ_{SSD} \mathbf{r'w}$) if

$$F^{(2)}_{\mathbf{r'v}}(t) \le F^{(2)}_{\mathbf{r'w}}(t) \quad \forall t \in \mathrm{R}$$

with strict inequality for at least one $t \in \mathrm{R}$. This relation is sometimes called strict second-order stochastic dominance because the strict inequality for at least one $t \in \mathrm{R}$. is required, see [9] for more details. Alternatively, one may use several different ways of defining the second-order stochastic dominance (SSD) relation: $\mathbf{r'v} \succ_{SSD} \mathbf{r'w}$ if and only if $\mathrm{E}u(\mathbf{r'v}) \ge \mathrm{E}u(\mathbf{r'w})$ for all concave utility functions u provided the expected values above are finite and strict inequality is fulfilled for at least some concave utility function, see for example [9].

We say that portfolio $\mathbf{w} \in W_M$ is SSD inefficient with respect to W_M if and only if there exists portfolio $\mathbf{v} \in W_M$ such that $\mathbf{r'v} \succ_{SSD} \mathbf{r'w}$. Otherwise, portfolio **w** is SSD efficient with respect to W_M. By analogy, portfolio $\mathbf{w} \in W_B$ is SSD inefficient with respect to W_B if and only if there exists portfolio $\mathbf{v} \in W_B$ such that $\mathbf{r'v} \succ_{SSD} \mathbf{r'w}$. This definition classifies portfolio $\mathbf{w} \in W_M$ or $\mathbf{w} \in W_B$ as SSD efficient if and only if no other portfolio from W_M or W_B is better (in the sense of the SSD relation) for all risk averse and risk neutral decision makers.

Since the decision maker may form infinitely many portfolios, the criteria for pairwise comparisons have only limited use in portfolio efficiency testing. To test whether a given portfolio **w** is SSD efficient, three linear programming tests were developed, see [6], [9], [17]. We formulate the tests for SSD efficiency with respect to W_M. However, one can easily rewrite it for W_B.

In our empirical section, we apply the Kuosmanen SSD efficiency test [6] based on majorization principle, see [5]. The test consists of solving two linear programs, in order to identify a dominating portfolio (if it exists) which is already SSD efficient. The difference between mean return of tested portfolio **w** and its dominating portfolio can be seen as a measure of SSD inefficiency of portfolio **w**. The higher the measure is, the greater improvement (in terms of mean returns) can be done by moving from **w** to $\mathbf{v}^*$, see [9] for more details. If a tested portfolio is SSD efficient then the measure is equal to zero.

4 EMPIRICAL STUDY

We consider monthly exceeded returns of ten US representative industry portfolios which represent n = 10 basic assets. The returns can be found in data library of Kenneth French. We consider 30 years period 1980 – 2010, that is T = 360 historical scenarios.

The first task is to determine the optimal weights of particular currencies for min-var portfolios following either the approach of Markowitz (no short selling) or Black (short selling up to the initial investment is allowed) on the basis of three distinct dependence/concordance matrices.

See Table 1 for review of all portfolios we deal with. In the same table we provide for each portfolio the mean exceeded return and classical standard deviation.

Table 1 Denotation of particular portfolios and their characteristics

Portfolio	Correlation (*R*)	Short selling	Mean exceeded return (%)	Stand. dev.
Π_M1	Pearson	No	0.613	3.525
Π _B1	Pearson	Yes	0.653	3.453
Π _M2	Spearman	No	0.618	3.534
Π _B2	Spearman	Yes	0.655	3.471
Π _M3	Kendall	No	0.626	3.562
Π _B3	Kendall	Yes	0.634	3.536

Following Table 1 we can see that all three min-var portfolios with possible short positions have higher mean exceeded return and smaller standard deviation than corresponding portfolios for the case of no short sellings. We will proceed with SSD portfolio efficiency testing of these 6 portfolios. It is well known, that portfolios with minimal standard deviation generally need not to be SSD efficient.

Since 6 considered min-var portfolios are constructed as portfolios with minimal risk we expect that they have relatively small mean exceeded returns. Therefore we choose the Kuosmanen test for SSD portfolio efficiency testing. If the tested portfolio is SSD inefficient, the Kousmanen test gives us information about SSD dominating portfolio with the highest mean exceeded return and the SSD inefficiency measure identifies the maximal possible improvement (in terms of mean exceeded returns) that can be done by moving from a min-var portfolio to better ones (in sense of SSD relation). The results of the Kuosmanen test for considered portfolios are summarized in Table 2 (for the correlation used for particular portfolios, as well as the short selling constraint see Table 1).

Table 2 SSD efficiency results of min-var portfolios

Port.	SSD efficiency	Mean exceeded return (%)	Measure of SSD inefficiency	Mean exceeded return of SSD dominating portfolio (%)
Π_M1	Yes	0.613	0.000	0.613
Π_B1	No	0.653	0.034	0.687
Π_M2	No	0.618	0.022	0.640
Π_B2	No	0.655	0.067	0.722
Π_M3	No	0.626	0.031	0.657
Π_B3	No	0.634	0.134	0.768

Table 2 shows us that only using Pearson correlation and no short positions the min-var portfolio is SSD efficient. All other min-var portfolios were classified as SSD inefficient and for every min-var portfolio exists some SSD dominating portfolio with higher mean return. Moreover, in the case of included short selling the SSD inefficiency measures are higher than in the other case, that is, portfolios with minimal measures of concordance in the Black model are more SSD inefficient than that in the Markowitz model. Moreover, in both cases, the Pearson min-var portfolios have smaller measures of SSD inefficiency than the other corresponding min-var portfolios. This finding is perhaps surprising because the Pearson min-var portfolios have also the smallest mean exceeded returns. The largest measures of SSD inefficiency are observed when considering Kendall measure of concordance. Therefore, we suggest using Pearson measure of concordance.

5 CONCLUSION

In this paper we have studied the SSD (in)efficiency of several portfolios created from the ten US industrial representative portfolios. These portfolios are minimizing risk, when the dependency matrix is build up on the basis of alternative concordance measures (namely, Pearson and Kendall measures of dependency).

We have observed that almost all min-var portfolios were classified as SSD inefficient. It means that all risk averse investors prefer some other portfolio than these min-var ones. The only exception is the case when Pearson measure of concordance is considered and short positions are not allowed. Portfolio identified in this case is the optimal choice of expected utility maximising problem for at least one concave utility function. Moreover, comparing SSD inefficiency measures of min-var portfolios, the best concordance measure is Pearson one, no matter if short positions are allowed or not. All these results can be of great value for portfolio managers in banks and other financial institutions. However, before making a final conclusion about the suitability of particular risk and dependency measures in portfolio theory also other measures of dependency should be examined. Alternatively, one can enrich the stochastic dominance analysis in several ways. For example, the issue of the first order stochastic dominance (FSD) can be employed, see [7] and [9]. Moreover, one can call for stability analysis of SSD efficiency, and follow [2] or [8], in order to study the impact of changes in scenarios on the efficiency classification.

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AN EVOLUTIONARY ALGORITHM FOR THE MIXED POSTMAN PROBLEM

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Abstract
The Graph augmentation is the key technique in solving several variants of the Postman Problem. The problem is transformed into a task of obtaining a minimal Eulerian multigraph. The mixed graph variant is NP-complete. We explore the possibilities of evolutionary optimization as a heuristic means to overcome the hardness of the problem. We propose an evolutionary algorithm that augments the original graph while aiming for minimal total cost. Individual solutions are represented as directed multigraphs. We devised a set of evolutionary operators specifically designed for the problem, including a repair operator to keep the solutions feasible. Experimental results are promising and show that the algorithm is efficient in comparison to known approximate algorithms concerned with the same task.

Keywords: *Genetic Algorithm, Mixed Chinese Postman Problem, Arc Routing*

JEL Classification: C44
AMS Classification: 90C15

1 INTRODUCTION

The postman problem is a well-known arc routing problem that consists of finding a minimum-cost tour of a graph traversing all its edges and/or arcs at least once. It is the optimization version of the Euler tour problem asking if there is a tour that traverses every edge of a graph exactly once [11]. The mixed problem is defined over an arbitrary connected mixed graph with nonnegative costs. In [8] Papadimitriou showed that the mixed problem is NP-complete. No polynomial optimal algorithm is known to solve it. Integer linear programs and various branching heuristics were given by Ralphs [12], Kappauf/Koehler, Grötschel/Win [2], Nobert/Picard [7], and Christofides [5]. Several approximate algorithms were shown to solve the problem in strictly polynomial time but in a sub-optimal manner including heuristics by Edmonds/Johnson [1], Frederickson [4], Raghavachari/Veerasamy [11], and others. All these algorithms share a common link: They augment the original graph into an Eulerian graph by copying and orienting some of the edges and arcs. A postman tour can be determined by finding an Eulerian tour in the augmented graph. We present a promising *evolutionary algorithm* that evolves a population of directed multigraphs as individual solutions to the *mixed postman problem*. Its performance is explored by an experiment.

2 GRAPH AUGMENTATION

A *directed graph* $G = (V, A)$ is a set of vertices V along with a multiset A of directed arcs defined over V. A *mixed graph* $G_M = (V, U, A)$ extends the definition with a multiset U of undirected edges. *Edge* $u = \{v, w\}$ is an unordered pair of vertices, whereas *arc* $a = (u, w)$ is an ordered pair of vertices that are connected in the graph. We expect each arc and edge to have a positive weight $w(e)$, $e \in U \cup A$, and we call this weight *the cost of edge*, or *the cost of arc*. In a multigraph, there can be more than one edge or arc between a pair of vertices. To solve a mixed postman problem, it is necessary to find a closed walk that contains every edge and arc of the graph. Augmentation based algorithms transform the original graphs into Eulerian multigraphs, which is the key and the hard step of the solution. The resulting multigraph is used to obtain a simple circuit containing every edge and arc of it once and only once. The circuit in multigraph corresponds to a closed walk in the original graph.

In the proposed evolutionary algorithm, the augmented multigraphs are directed. To augment the original mixed graph, some arcs and edges may be copied and all edges must be oriented. As defined by König [6], a connected directed graph is Eulerian if and only if for each vertex of the graph, the outdegree of the arcs adjacent to it equals the indegree. The goal is to find a minimal-cost Eulerian augmented graph as to minimize the cost of the postman tour obtained from it. The cost of the graph is given by sum of costs of all its arcs, counting parallel arcs multiple times. The cost of the postman tour equals the cost of the Eulerian multigraph.

3 EVOLUTIONARY AUGMENTATION ALGORITHM

The evolutionary algorithm is used to augment the original mixed graph into a population of Eulerian multigraphs. These graphs are directed, which is the key design attribute of the algorithm. Every solution (population individual) is kept feasible at the end of a generation. We devised a special set of domain specific evolutionary operators, yet they are not neutral and cause the solution to become infeasible. We designed a repair operation to counteract this. The resulting graph of this operation is always feasible - Eulerian.

3.1 Definition and representations

Primarily three constituents define an evolutionary algorithm: the representation of the individual solution, the fitness function, and a set of evolutionary operators defined over individuals. The *individual* represents an augmented and eventually directed Eulerian multigraph G_D. The *individual* $Y = (G_M, D)$ consists of a reference to the original mixed graph $G_M = (V, U, A)$ and a set D of adjacency descriptors. Let u and v be any two adjacent vertices of the mixed graph G_M that are expressed as some integer indices, where $u < v$. Then $D = \{d_{uv} \mid u < v \wedge [(u,v) \in A \vee (v,u) \in A \vee \{u,v\} \in U]\}$. For each applicable pair of u and v, the *adjacency descriptor* $d_{uv} = (n_{uv}, n_{vu})$ denotes the count of occurrences n_{uv} of the up arc (u,v) and the count of occurrences n_{vu} of the down arc (v,u) in G_D. Such a structure is memory-saving and allows for simple copying and reorienting of the arcs. The reference to the original mixed graph is necessary, so that original arcs are preserved – they pose as the lower boundary for n_{uv} and n_{vu}. Also, the number of original edges must be conserved and represented by free arcs of either direction. Let o_{euv}, o_{uv}, o_{vu} be the number of original edges, up arcs and down arcs between u and v in G_M. The following inequalities must be satisfied at all times: $o_{euv} + o_{uv} + o_{vu} \le n_{uv} + n_{vu}$, $o_{uv} \le n_{uv}$ and $o_{vu} \le n_{vu}$.

Individual graphs in population are kept Eulerian and such graphs may be summed up for total cost of the postman tour very easily. Then the fitness (to be maximized) of an individual is reciprocal to the total cost of the graph (to be minimized). There are four basic groups of evolutionary and hybrid operators used in the algorithm: Mutation and local optimization operators, recombination, repair operator. Additionally, these are aided by the reference solution reinsertion operators and the repopulation operator.

3.2 Initialization and repair operator

A reasonably diverse initial population must be generated for the recombination operation to be truly functional and the evolution robust. In practical terms, this may only be done by random generation. Moreover, we need every graph in the population to be Eulerian. We designed an algorithm that takes the input graph and applies random augmenting modifications to it until the graph is Eulerian. Before this is done, all edges of the mixed graph are oriented at random. The graphs may become too heavy under the additional paths added to them; the reduction step removes any cycles found in the augmentation - such a graph will remain Eulerian. The augmentation and reduction steps form the *repair operator*. The augmentation procedure will randomly select matching pairs of vertices with unequal indegree and outdegree and connect them by random or shortest directed paths (copying arcs in the process). The cycle removal may be implemented by depth-first searches. The repair operator is executed for every unfeasible solution at the end of each evolution cycle step.

3.3 Recombination

A probability *q*-tournament is used to select pairs of parent graphs for a crossover operation. For every pair of parents, a pair of children is generated. The operation selects a random connected set of vertices of random size. Then it exchanges between the (children) individuals the arcs between the vertices of this set. This operation does not keep feasibility, which will be corrected later by the repair operator. The idea is to propagate good local solutions.

3.4 Mutations and local optimization

The mutation is applied at every generation, to a randomly selected subset of individuals. The operator is unary. We devised three random mutation techniques and one hybrid operation that incorporates a local optimization technique. The algorithm will randomly select one of the mutations per individual:

- The hybrid *path rerouting* operation selects a random walk in the individual's graph augmentation, and if it is longer than the shortest possible path between the end vertices, it is replaced by it. This operation directly reduces the total weight of the graph arcs, increases the individual fitness and it does not affect feasibility.
- *Rotation* mutation picks a random subset of arcs from the graph that may be rotated, and changes their direction, $(u,v) \to (v,u)$. To pick an arc, it must come from an original edge. It causes possible infeasibility of feasible solutions by asymmetry of vertices.
- *Arc duplication mutation* repeatedly selects random pairs of adjacent vertices and inserts arcs between them, in random but permitted direction. If there was an edge between the vertices originally, either direction is permitted. The operation causes possible infeasibility.
- *Arc removal mutation* repeatedly deletes randomly selected copies of arcs that are spare, i.e. the condition $o_{euv} + o_{uv} + o_{vu} < n_{uv} + n_{vu}$ must be satisfied (as inequality) prior to the selection with u and v as vertices incident with the arc.

3.5 Reference individuals and concept of museum

Two methods of individual persistence were devised. Elitism is used in its usual form. One or more elite individuals from previous populations are retained for the new one. The second concept we named museum; it is a simple stack that contains those individuals that broke through in their generations. Every time evolution produces a solution more fit than overall maximum, it will be stored at the top of the museum. Occasionally, an individual will be selected from the museum and reinserted into population. The museum can be used in latter analysis of the evolution process just like a history book.

The museum concept offers reinsertion of reference material, so that the population keeps on converging. The solution to the augmentation part of the postman problem may be obtained by solving an integer linear program. It can be assumed that the integer solution is somewhere near the solution of its relaxation. We use this idea and calculate optimal solution of the linear relaxation of the original problem in parallel to the evolutionary cycle. When obtained, we save this solution and insert into the population sooner or later (it may be reinserted multiple times in the evolution). When inserting into population, arbitrary rounding is made and the graph is submitted to repair; the resulting Eulerian graph is then placed in the population.

4 EXPERIMENTAL RESULTS

The evolutionary algorithm EvoD was compared to a mixed strategy consisting of several well-known approximation algorithms that use deterministic heuristics:

- the original method by Edmonds and Johnson [1],
- MIXED 2 method by Frederickson [4],

- Modified MIXED algorithm by Raghavachari and Veeresamy [11],
- Modified, respectively improved versions of MIXED 1 and MIXED 2 algorithm by Pearn and Liu [10], respectively Pearn and Chou [9].

Within the mixed strategy, we executed each of these algorithms and picked the best solution. The evolutionary algorithm EvoD was executed 5 times and an average of the retrieved solutions was calculated to overcome its stochastic nature.

A wide experiment was devised. We implemented a mixed graph generator that used models of Erdős & Rényi [3], Watt & Strogatz [13] and our own custom model. The undirected graphs were then processed and randomly modified into mixed and connected graphs by means of a try & error approach. The graph generator produced 1698 random mixed graphs with 6-200 vertices, 9-1256 edges and arcs. The evolutionary algorithm EvoD was set-up with population size of 26 individuals and it terminated after 2000 generations of stagnation. The test results and problem sizes grouped by number of vertices are shown in Table 1.

Table 1 Comparison of results by EvoD and the strategy of approximate methods

Number of vertices	Number of graphs	EvoD wins		MIXED wins		Tied results	
		absolute	relative	absolute	relative	absolute	relative
0-19	89	19	21,35%	0	0,00%	70	78,65%
20-39	148	67	45,27%	13	8,78%	68	45,95%
40-59	168	99	58,93%	20	11,90%	49	29,17%
60-79	180	118	65,56%	28	15,56%	34	18,89%
80-99	288	137	47,57%	41	14,24%	110	38,19%
100-119	230	98	42,61%	34	14,78%	98	42,61%
120-139	154	105	68,18%	33	21,43%	16	10,39%
140-159	154	98	63,64%	36	23,38%	20	12,99%
160-179	163	79	48,47%	57	34,97%	27	16,56%
180-200	124	69	55,65%	50	40,32%	5	4,03%
Totals:	**1698**	**889**	**52,36%**	**312**	**18,37%**	**497**	**29,27%**

As Table 1 shows, the EvoD won 889 cases and broke even in another 497 tests out of 1698. In total it produced better or comparable results in 1386 cases (81,63%). It is apparent that the performance of the evolutionary algorithm does not drop with problem size relative measures of the contest.

5 CONCLUSION

The experimental results confirm that the evolutionary approach is competitive and will deliver good results in comparison to available heuristic alternatives. Given enough time and population size, there is no obstacle that would keep the algorithm from finding global optima. One of the setbacks of evolutionary computation techniques may be their runtime. However, for one evolutionary step EvoD is polynomial and thus scalable. Furthermore, the algorithm is easily parallelized. The directed multigraph representation of the individual solution makes it possible to modify the method easily to support asymmetric costs of edges and allow for solving the less covered and NP-hard Windy Postman Problem, a subject we shall cover in future publications.

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MULTI-CRITERIA EVALUATION OF ALTERNATIVES APPLIED TO THE MOBILE PHONE TARIFFS IN COMPARISON WITH MONTE CARLO SIMULATION RESULTS

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Abstract
Simulation modeling and multi-criteria evaluation of alternatives are two different principles of mathematical methods connected with the operational research. Monte Carlo simulation tries to iteratively evaluate the deterministic model by using random inputs. Methods of multi-criteria evaluation of alternatives use given inputs to find (usually) the order of the alternatives. This order is influenced mainly by weights of the criteria. In this article we try to find the optimal mobile phone tariff for the given employers by selected multi-criteria evaluation of alternatives method and compare the results with the Monte Carlo simulation model for the same employers. We would like to answer the question if it is possible to construct and solve the multi-criteria model with adjusted weights to find the same results as simulation model does.

Keywords: *Multi-criteria Evaluation of Alternatives, Monte Carlo Simulation, Mobile Phone Tariffs*

JEL Classification: C44, C15
AMS Classification: 91B06, 65C05

1 INTRODUCTION

Simulation methods belong to the suitable instruments that can be used in the real world situations to better understand the reality or to make a responsible decision. Simulation nowadays means a technique for imitation of some real situations, processes or activities that already exist in reality or that are in preparation – just to create a computer model [1]. The reasons for this are various: to study the system and see how it works, to find where the problems come from, to compare more model variants and select the most suitable one, to show the eventual real effects of alternative conditions and courses of action, etc. Simulation is used in many contexts, including the modeling of natural or human systems in order to gain insight into their functioning (manufacturing, automobile industry, logistics, military, healthcare, etc.), simulation of technology for performance optimization, safety engineering, testing, training and education.

In economy we must face a lot of decisions that have to be made, and pay a lot of money afterwards often without knowing whether we have done right or wrong. When everything is given, the solution or decision can be based on the common sense or on the solution of some mathematical model. But the problem is that a lot of things not only in economy are not certain – especially when we think about money spent for phoning. People are usually able to describe the expenses related to their calls "something between 200 and 500 crowns per month" or "350 crowns at a medium". Although it seems to be vague, inaccurate and insufficient, with some knowledge of statistical distributions we are able to use given information and even make a decision or recommendation via Monte Carlo simulation model. On the other hand there are some methods for decision-making that also can help in this situation – and here the preferences must be specified for example by the weights of the criteria. But is it possible to use multi-criteria evaluation of alternative methods to obtain the same results as from the simulation model?

We will describe the simulation model and we compare the results obtained from simulation in MS Excel with the results taken from the static decision-making model when the TOPSIS and ELECTRE III. methods are used.

2 DATA AND METHODS

Before we start the analysis we have to select the alternatives (mobile operators' tariffs), the criteria and the distributions for the random variables generation. Our analysis is aimed at the specific situation – to find the best tariff for one employee of the Executive Board of the Czech Union for Nature Conservation to minimize the costs of telephone calls. The entire model for more employees has been created in the diploma thesis [5] where all (69 possible) the mobile operators' tariffs and their data are described.

The problem occurs in the case when we don't know preferences of user in any form. Also in such case one solution of this problem is a simulation of weights. We randomly generate weight vector and use it as input for multi-criteria evaluation methods. Then we can observe changing alternatives that are at the first places. As we would like to compare the results with the order of the tariffs created by the TOPSIS and ELECTRE, we shortly describe also principles of these methods.

2.1 Multi-criteria Evaluation of Alternatives Methods

Multi-criteria evaluation of alternatives belongs to the category of discrete multi-criteria decision making models where all the alternatives ($a_1, a_2, \ldots, a_p$) and criteria ($f_1, f_2, \ldots, f_k$) are known. To solve this kind of model it is necessary to know the preferences of the decision maker. These preferences can be described by aspiration levels (or requirements), criteria order or by the weights of the criteria. We may find a lot of different methods [2], [3], [4], the two that we use are TOPSIS and ELECTRE III.

TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution)

The output provided by TOPSIS is a complete arrangement of possible alternatives with respect to the distance to both the ideal and the basal alternatives incorporating relative weights of criterion importance. The required input information includes decision matrix **Y** and weight vector **v**. In addition, in the same way as in the WSA an assumption of maximization of all the criteria is true (otherwise it is necessary to make an appropriate transformation). This decision-making approach can be summarized in the following steps:

- normalize the decision matrix according to Euclidean metric:

$$r_{ij} = \frac{y_{ij}}{\sqrt{\left(\sum_{i=1}^{p}(y_{ij})^2\right)}}, \quad i = 1, 2, \ldots, p, \;\; j = 1, 2, \ldots, k,$$

- calculate the weighted decision matrix $\mathbf{W} = (w_{ij})$: $w_{ij} = v_j \cdot r_{ij}$,
- from the weighted decision matrix **W** identify vectors of the hypothetical ideal and basal alternatives over each criterion: $\mathbf{H} = (H_1, H_2, \ldots, H_k)$ and $\mathbf{D} = (D_1, D_2, \ldots, D_k)$, where $H_j = max_i\, w_{ij}, \;\; j = 1, 2, \ldots, k$ and $D_j = min_i\, w_{ij}, \;\; j = 1, 2, \ldots, k$,
- measure the Euclidean distance of every alternative to the ideal and to the basal alternatives over each attribute:

$$d_i^+ = \sqrt{\sum_{j=1}^{n}\left(w_{ij} - H_j\right)^2} \text{ and } d_i^- = \sqrt{\sum_{j=1}^{n}\left(w_{ij} - D_j\right)^2},$$

- for each alternative determine the relative ratio of its distance to the basal alternative:

$$c_i = \frac{d_i^-}{d_i^+ + d_i^-},$$

- rank order alternatives by maximizing ratio c_i.

ELECTRE III (ELimination Et Choix Traduisant la REalité / Elimination and Choice Expressing Reality)

A sophisticated improvement introduced by this method in comparison with the other ELECTRE methods would be the use of pseudo-criteria instead of true-criteria. ELECTRE III ensures placement of the alternatives into groups according to their performance value. Alternatives inside one group are usually valued identically; whereas groups themselves are interrelated by the means of preference relations. Similarly to AGREPREF method, ELECTRE III starts with comparing each pair of alternative a_i and a_j and determining two sets of criteria indexes for each such pair: I_{ij} – contains criteria, over which a_i is preferred to a_j, I_{ji} – contains criteria, over which a_j is preferred to a_i.

Then preference levels s are defined as $s_{ij} = \sum_{h \in I_{ij}} v_h$, and $s_{ji} = \sum_{h \in I_{ji}} v_h$. In this method, a decision maker does not need to enter the threshold values (as in the case of ELECTRE I.), instead they are gradually generated in the course of the solution process. First, the highest level of preference c^0 is found in the preference matrix **S** = (s_{ij}): $c^0 = max\{s_{ij};\ a_i, a_j \in A\}$, where A denotes a set of alternatives.

The first threshold value can be defined as the second highest value from the matrix **S**: $c^1 = max\{s_{ij};\ a_i, a_j \in A, s_{ij} < c^0\}$. Then we set p_i^1 – a number of alternatives that a_i is preferred to with respect to the preference threshold value c^1, and q_i^1 – a number of alternatives that are preferred to a_i with respect to the preference threshold value c^1. The alternatives are then assigned to the relevant groups according to the d-indicator: $d_i^1 = p_i^1 - q_i^1$. Therefore a subset A^1 includes alternatives with the highest d value: $A^1 = \{a_i;\ \max_i d_i^1\}$. In the case the subset A^1 contains more than one alternative, it is essential to check the possibility of ordering the alternatives.

Subsequently, subsets A^2, A^3, … are determined identically. The procedure continues until one of the following situations occurs: A^r is a single alternative subset or $c^r = 0$ and the remaining alternatives are indifferent and constitute a group.

2.2 Monte Carlo Simulation

Monte Carlo simulation (or technique) is closed to statistics as it is a repeated process of random sampling from the selected probability distributions that represent the real-life processes [7]. On the basis of the existed information we should select the type of probability distribution that corresponds to our expectations and define all the parameters for.

The problem of some economic models is the lack of the information – especially in the retail sector sometimes only managers themselves know how the process works, what the typical number of customers during a period is etc. In this kind of situations we cannot use basic statistical or mathematical models as we do not have the strict or real data. That is why Monte Carlo simulation can help as it uses random variables from different distributions. The usage of MS Excel and Crystal Ball for the mobile phone tariffs is described in [6]. This kind of simulation was used also in the diploma work [5] to find the best tariff. But it is possible to use it also to generate the weights of the criteria – or better to say generate the points for each criterion and then calculate the weights using the Point method [3].

2.3 Selection of the Mobile Operators' Tariffs

All the information about the mobile operators' tariffs we have taken from the diploma work [5]. The description of the telephone calls of the selected person is following: during the usual month her duration time of all calls is about 400 minutes (it was used in transformation of the criteria), she call mainly during a week, the average length of one call is about 4 minutes. The number of calls during a day has been approximated by Poisson distribution with parameter λ=4.88, but this information is not important for the multi-criteria methods.

We have created 6 types of criteria to compare 69 tariffs. For Monte Carlo simulation all data in rough form can be used. However for multi-criteria decision-making we need to transform data in relevant form. We use three obvious criteria: fixed payment tariff (minimal), price for 1 minute calling in own net (minimal) and price for 1 minute calling in other net (minimal). The fourth criterion is the number of free minutes (maximal). As we search the best tariff for employee who calls about 400 minutes per month e.g. tariff with 2400 free minutes is indifferent to tariff with 500 free – in such case 2400 free minutes is not advantage in comparison with 500 free minutes. Therefore we transformed given data for free minutes: free minutes for analysis are defined as min{500, free minutes of tariff}. In the end we included two more criteria: advantages in own net (maximal) and advantages in other net (maximal), both criteria are evaluated by positive points with maximal value 10. In this situation we have founded out that from the 69 tariffs only 33 are non-dominated, so we can compare only these tariffs. The only problem is how to choose weights. It is clear that for the given situation the most important criterion is the fixed payment per month and afterwards the price for 1 minute calling. But as the preferences are not clear enough we have used the random generation of weight vector.

3 RESULTS

The simulation model of the described situation showed [5] that the best tariffs are from the T-mobile operator called "Podnikatel Plus 450" and "Podnikatel Plus 700". Our calculation of TOPSIS and ELECTRE III and the weight vector generation show that the weights are crucial for the results. From the weights simulation we have detected that it is not possible to find a weight vector for the "Podnikatel Plus 700" tariff to be on the first place. The best results we have obtained by ELECTRE III method where there exist weights for all the criteria for the tariff "Podnikatel Plus 450" to be the best one or for both to be on the second and third position. Via the TOPSIS method it is not possible to have this tariff on better than fourth place. Table 1 shows the best results.

Table 1 The best positions of the selected tariffs and the weight vectors

trial	Fixed payment	Adv. own	Adv. other	Free min.	Price 1 min own	Price 1 minute other	Order Pod.Plus 450 ELECTRE III / TOPSIS	Order Pod.Plus 700 ELECTRE III / TOPSIS
1	0,403	0,016	0,016	0,080	0,242	0,242	**1** / 15	5 / 8
2	0,750	0,013	0,016	0,187	0,017	0,017	2 / 6	**3** / **2**
3	0,688	0,006	0,0229	0,158	0,062	0,062	2 / 5	**3** / 3
4	0,664	0,032	0,017	0,042	0,122	0,122	2 / **4**	4 / 10

4 CONCLUSION

From the simulation we can see that there exist weight vectors that provide similar solution (the set of best alternatives) as in the case of Monte Carlo simulations. In the case of TOPSIS usage it is possible to show that there exists no such weight vector to "Podnikatel Plus 450" or "Podnikatel Plus 700" tariffs being the best alternative. We can also see that ELECTRE III. method provides more similar results as Monte Carlo simulation in comparison to TOPSIS method. In this situation the weights are crucial and so we cannot guarantee that the multi-criteria methods are able to choose the best tariff for the given conditions. The Monte Carlo simulation is therefore the better way of solving this kind of problem.

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TRADE-OFF THE ACCURACY FOR COMPUTATIONAL TIME IN APPROXIMATE SOLVING TECHNIQUE FOR THE P-MEDIAN PROBLEM

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Abstract
This paper deals with an approximate approach to the p-median problem. Real instances of the problem are characterized by considerably big number of possible service center locations, which can take the value of several hundreds or thousands. Attempts at exact solving of these problems using a location-allocation model often fail due to enormous computational time or huge memory demands. Presented covering approach uses a lower approximation of a common distance by some pre-determined distances given by so-called dividing points. Deploy-ment of the dividing points influences the accuracy of the solution. To improve this approach, we have developed a sequential method of dividing point deployment. The main goal of this study is to show how the number of dividing points influences the accuracy of the solution in comparison to saved computational time.

Keywords: *p-median problem, approximate covering model, lower bound, sequential method*

JEL Classification: C61
AMS Classification: 90C27

1 INTRODUCTION

The public service system structure [5], [7] is formed by deployment of limited number of service centers and the associated objective is to minimize social costs which are proportional to distances between served objects and the nearest service centers. Mathematical models of the public service system structure problem are often related to the p-median problem, where the number of served customers takes the value of several thousands and the number of possible locations can take this value as well. The number of possible service center locations seriously impacts the computational time [6]. The necessity of solving large instances of the p-median problem has led to the approximate approach which enables us to solve real-sized instances in admissible time [2], [3]. This approach pays for shorter computational time or smaller computer memory demand by a loss of accuracy. The accuracy can be improved by a convenient determination of so-called dividing points which are used in the objective function approximation. Suggested sequential approach to the dividing points determination proved to be suitable not only for obtaining a good solution of the problem, but also for gaining a good lower bound of the unknown optimal solution. The experiments [4] with solving very large instances by the approximate approach gave enormous precise results, which differed from the exact solution by less than one percent. Unfortunately, the computational time necessary for the lower bound determination was very high. In this contribution, we focus on this phenomenon and suggest a "trade-off" the accuracy for shorter computational time.

2 SEQUENTIAL APPROACH TO THE P-MEDIAN PROBLEM

The p-median problem is formulated as a task of determination of at most p network nodes as facility locations so that the sum of distances between each node and the nearest facility is minimal. To describe the p-median problem on a network we denote J a set of served nodes, similarly, we denote I a set of possible center locations. The network distance between a possible location i and a customer j from J is denoted as d_{ij}. The basic decisions in any solving process of the p-median problem concern location of centers at network nodes from the set I. To model these decisions at particular nodes, we introduce a zero-one variable $y_i \in \{0,1\}$, which takes the value of 1 if a center should be located at the location i, and which takes the value of 0 otherwise. The discussed approximate approach is based on a relaxation of the assignment of a service center to a customer. The distance between a customer and the nearest facility is approximated unless the facility must be assigned. To obtain an upper or lower bound on the original objective function, the range $<0, max\{d_{ij}: i \in I, j \in J\}>$ of all possible distances is partitioned into $r+1$ zones. The zones are separated by a finite ascending sequence of dividing points $D_1, D_2, \ldots, D_r$ chosen from the sequence where $0 = D_0 < D_1$ and $D_r < D_m = max\{d_{ij}: i \in I, j \in J\}$. The zone k corresponds with the interval $(D_k, D_{k+1}>$. The length of the k-th interval is denoted by e_k for $k = 0, \ldots, r$. In addition, an auxiliary zero-one variables x_{jk} for $k = 0, \ldots, r$ are introduced. The variable x_{jk} takes the value of 1 if the distance of the customer $j \in J$ from the nearest located center is greater than D_k and this variable takes the value of 0 otherwise. Then the expression $e_0x_{j0} + e_1x_{j1} + e_2x_{j2} + e_3x_{j3} + \ldots + e_rx_{jr}$ is an upper approximation of the distance d_{j*} from customer j to the nearest located center. If the distance d_{j*} belongs to the interval $(D_k, D_{k+1}>$, it is estimated by the upper bound D_{k+1} with a maximum possible deviation e_k. Similarly to the covering model, we introduce a zero-one constant a_{ij}^k for each triple $<i, j, k> \in I \times J \times \{0, \ldots, r\}$. The constant a_{ij}^k is equal to 1 if and only if the distance d_{ij} between the customer j and the possible location i is less or equal to D_k, otherwise a_{ij}^k is equal to 0. Then a covering-type model can be formulated as follows:

$$Minimize \quad \sum_{j \in J} \sum_{k=0}^{r} e_k x_{jk} \tag{1}$$

$$Subject\ to: \quad x_{jk} + \sum_{i \in I} a_{ij}^k y_i \geq 1 \quad for\ j \in J\ and\ k = 0, \ldots, r \tag{2}$$

$$\sum_{i \in I} y_i \leq p \tag{3}$$

$$x_{jk} \geq 0 \quad for\ j \in J\ and\ k = 0, \ldots, r \tag{4}$$

$$y_i \in \{0,1\} \quad for\ i \in I \tag{5}$$

The objective function (1) gives the upper bound of the sum of original distances. The constraints (2) ensure that the variables x_{jk} are allowed to take the value of 0 if there is at least one center located in radius D_k from the customer j. The constraint (3) limits the number of located facilities by p. To obtain a lower bound of the objective function value of the optimal solution of the original problem, we realize that the interval $(D_k, D_{k+1}>$ given by a pair of succeeding dividing points contains exactly the elements $D_k^1, D_k^2, \ldots, D_k^{r(k)}$ of the sequence $d^0 < d^1 < \ldots < d^m$. The elements are strictly greater than D_k and less than D_{k+1}. If a distance d between a customer and a possible center location belongs to the interval $(D_k, D_{k+1}>$, then the maximum deviation of d from the lower estimation D_k^1 is $D_{k+1} - D_k^1$. As the variable x_{jk} from the model (1)-(5) takes the value of 1 if the distance of the customer $j \in J$ from the nearest located center is greater than D_k and this variable takes the value of 0 otherwise, we can redefine the zone coefficients e_k in accordance to $\underline{e}_0 = D_0^1 - D_0$ and $\underline{e}_k = D_{k+1}^1 - D_k^1$ for $k = 1, \ldots, r$. Then the expression $\underline{e}_0x_{j0} + \underline{e}_1x_{j1} + \underline{e}_2x_{j2} + \underline{e}_3x_{j3} + \ldots + \underline{e}_rx_{jr}$ is a lower approximation of the d_{j*}, which corresponds to the nearest distance of the served node j from the nearest located center i. The redefined objective function value of the optimal solution gives the lower bound of the original problem [1]. Having solved both problems, the better of two obtained solutions concerning the original objective function gives the resulting solution.

The number r and deployment of the dividing points influence the size of the covering model (1)-(5) and the accuracy of the result. The dividing points can be chosen only from the set of values $d^0 < d^1 < \ldots < d^m$ of the distance matrix $\{d_{ij}\}$, where $D_0 = d^0$ and $D_m = d^m$. Let the value d^h have a frequency N_h of its occurrence in the matrix $\{d_{ij}\}$. In the suggested approach, we start from a hypothesis that the distance d^h from the sequence $d^0 < d^1 < \ldots < d^m$ occurs in the resulting solution n_h times and that is why the deviation of this distance from its approximation encumbers the total deviation proportionally to n_h. The distance d between a customer and the nearest

located center can be only estimated taking into account that it belongs to the interval $(D_k, D_{k+1}>$, which contains only values $D_k^1, D_k^2, \ldots, D_k^{r(k)}$.
When a lower bound is computed, the expression $\underline{e}_0 x_{j0} + \underline{e}_1 x_{j1} + \underline{e}_2 x_{j2} + \underline{e}_3 x_{j3} + \ldots + \underline{e}_r x_{jr}$ is used as a lower approximation of d_{j^*}. Here $\underline{e}_0 = D_0^1 - D_0$ and $\underline{e}_k = D_{k+1}^1 - D_k^1$ for $k = 1,\ldots, r$. If a relevancy n_h of each d^h is given, we could minimize the deviation using other series of dividing points obtained by solving the following problem:

$$\textit{Minimize} \quad \sum_{t=0}^{m-1}\sum_{h=t}^{m-1}(d_{h+1} - d_{t+1})n_{h+1}\underline{z}_{th} \tag{6}$$

$$\textit{Subject to}: \quad \underline{z}_{t(h+1)} \le \underline{z}_{th} \quad \textit{for } t = 0,\ldots,m-1 \textit{ and } h = t,\ldots, m-1 \tag{7}$$

$$\sum_{t=0}^{h}\underline{z}_{th} = 1 \quad \textit{for } h = 0,\ldots,m-1 \tag{8}$$

$$\sum_{t=1}^{m-1}\underline{z}_{tt} = r \tag{9}$$

$$\underline{z}_{th} \in \{0,1\} \quad \textit{for } t = 0,\ldots,m-1 \textit{ and } h = t,\ldots,m \tag{10}$$

If the distance d^h belongs to the interval starting with a possible dividing point d^t then the decision variable $\underline{z}_{th}$ takes the value of 1. Link-up constraints (7) ensure that the distance d^{h+1} can belong to the interval starting with d^t only if each distance between $d^{h+1}$and d^t belongs to this interval. Constraints (8) assure that each distance d^h belongs to some interval and the constraint (9) enables that only r dividing points will be chosen. After the problem (6)-(10) is solved, the nonzero values of $\underline{z}_{tt}$ indicate the distances which correspond with dividing points for lower bounding process. The sequential zone adjustment method is based on the idea of making the estimation of individual distance d^k relevancy more and more accurate. The distance relevancy also means here a measure of our expectation that this distance value is the distance between a customer and the nearest located service center, but this estimation is improved step by step [3].
The initial relevancy estimation is related to the frequency N_h of d^h occurrence. To make the relevance estimation more realistic, we suggest to make use of the optimal solution of the approximate problem described by (1)-(5) and (6)-(10). Assuming that the optimal solution of the approximate problem is known, especially knowing which y_i is equal to one, only the associated rows of the matrix $\{d_{ij}\}$ are taken as so-called active rows. Then each column of this matrix is processed and the minimal value over the active rows is included into the set of relevant distances and their occurrence frequencies. Thus a new sequence of distance frequencies n_k is obtained. The first and last distances d^0 and d^m are taken from the former sequence. This new sequence is used in the problems (1)-(5) and (6)-(10) and the optimal solution of this problem yields a new set of dividing points. Based on this new dividing point set, the sequence of zones is readjust and for new zone sequence the approximate problems are resolved. This process can be repeated until the stopping criterion is met.

3 TRADE-OFF THE ACCURACY FOR LOWER BOUND COMPUTATIONAL TIME

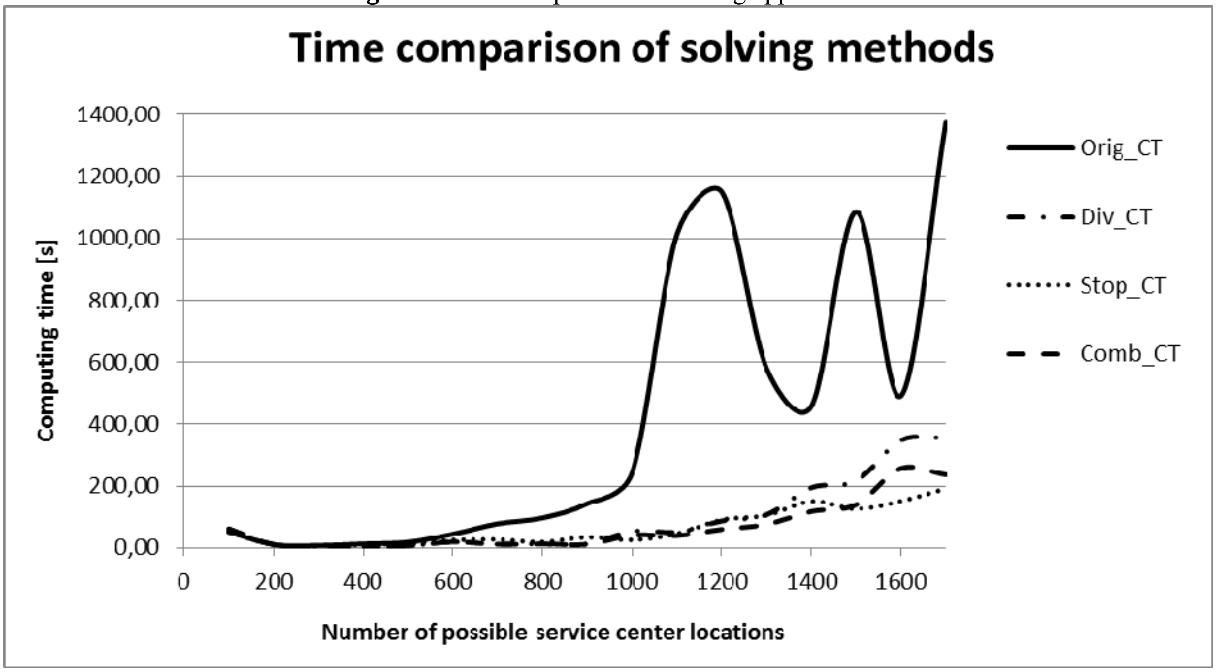
The above approach was tested on the pool of benchmarks [4], whose size varied from 1 to 17 hundreds of possible center locations. The number of dividing points was set to the value of 20 in all solved instances and also the same stopping criterion was used to terminate the main loop, which performs individual iterations. The stopping criterion combined conditions of reaching maximum number 10 of performed iterations and performing only one iteration without improving the lower bound. As shown in the Table 1 (see the row Orig_Gap) and on the Figure 1 (see the curve "Orig_CT"), this approach proved to be very efficient as concerns the accuracy, but the demanded computational time was unstable and extremely high. This phenomenon evoked to us an idea to make a "trade-off" between the excellent accuracy and the high computational time of the approach. We focus on two possible ways within this contribution.

The first one is based on reducing the prescribed number of dividing points. The reduced number of dividing points causes a loss of accuracy, but, on other hand, it can reduce the size of the problem (1)-(5), what can be followed by computational time reduction. The second way consists of adjustment of the stopping criterion, where reaching of a tolerable deviation below two percents of the best found solution is taken as the third condition for the loop termination.

Table 1 Average gaps between the best found solutions and the associated lower bounds in percents of the best found solution

$\|I\|$	100	300	500	700	900	1100	1300	1500	1700
Orig_Gap	4.20	2.12	0.75	0.25	0.10	0.07	0.06	0.05	0.13
Div_Gap	9.01	7.52	5.43	3.29	2.21	1.38	1.44	1.22	1.34
Stop_Gap	4.20	2.12	0.80	0.36	0.20	0.17	0.24	0.21	0.32
Comb_Gap	9.01	7.52	5.43	3.29	2.37	1.39	1.45	1.28	1.47

Figure 1 Time comparison of solving approaches



The both approaches have been tested on the pool of benchmarks and the associated results are plotted in Tab. 1 and Fig. 1. The table 1 contains average gaps the best found solutions and the associated lower bounds in percents of the best found solution values. Each column corresponds to size of 6 benchmarks, which were solved by the original approach (Orig_), the first (Div_) and second (Stop_) approaches of the suggested ones and by the approach, where both adjustments were applied (Comb_). The associated rows are denoted by the corresponding prefixes and this notation is also used for the associated curves in Fig. 1, where dependences of average computational times (CT) on the benchmark sizes are depicted.

4 CONCLUSIONS

Comparing the reported results, we can conclude the attempts at the trade-off by stating that both suggested approaches proved to be successful in considerable reduction of computational time under small loss of accuracy. It can be found that the second approach (Stop_) is better than the first one (Div_) and, in addition, it yields better results than the combined approach.

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FINANCIAL LIQUIDITY MANAGEMENT IN RELATION TO RISK SENSITIVITY: POLISH FIRMS CASE[1]

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Abstract

Liquid assets in firm are maintained for risk reduction purposes. The basic financial purpose of an enterprise is maximization of its value. Liquid assets management should also contribute to realization of this fundamental aim. The enterprise value maximization strategy is executed with a focus on risk and uncertainty. This article presents the consequences that can result from operating risk that is related to liquid assets management policy. An increase in the level of liquid assets in a firm increases both net working capital requirements and the costs of holding and managing working capital. Both of these decrease the value of the firm. But not always it works in the same way, it depends on risk sensitivity. Collected data shows how the Polish firms liquidity management model works in emerging markets reality. In the paper the relation between liquid levels and risk sensitivity is illustrated by empirical data from Polish firms.

Keywords: *liquidity, cost of capital, firm value*

JEL Classification: G32, G31, D24
AMS Classification: 90A09

1 INTRODUCTION

The hypothesis of the paper is presumtion that higher risk shown by beta coefficient, should results with more flexible and more conservative liquid assets strategies. Financing of the liquid assets has its cost depending on risk linked with liquid assets strategies used by the financed firm. If we have higher risk, we will have higher cost of financing (cost of capital) and as result other firm value growth. There are no free lunches.

Cost of financing of liquid assets depends on kind of financing, next on level of liquid assets in relation to sales and last but not least management risk aversion.

According to kind of financing we have three strategies:

- aggressive strategy with the most risky but the cheapest, mainly short-term financing,
- compromise strategy with compromise between risk and costs of financing and
- conservative strategy with the most expensive long-term financing and with the smallest level of risk.

Choosing between various levels of liquid assets in relation to sales, we use one from three strategies:

- restrictive strategy when management use the most risky but the cheapest, the smallest as possible, level of liquid assets,
- moderate strategy when management moderate between risk and costs of holding liquid assets, and
- flexible strategy when management use the most expensive and rather high levels of liquid assets wanting to hedge the firm before risk of shortage of liquid assets.

Risk aversion depends on position of the firm in its business branch. If the risk aversion should be higher, then more smart is to choose more flexible and more conservative solutions to have better results. It works in opposite direction also, the safe firms with near to monopoly positions can use more restrictive and more aggressive strategies to have better results.

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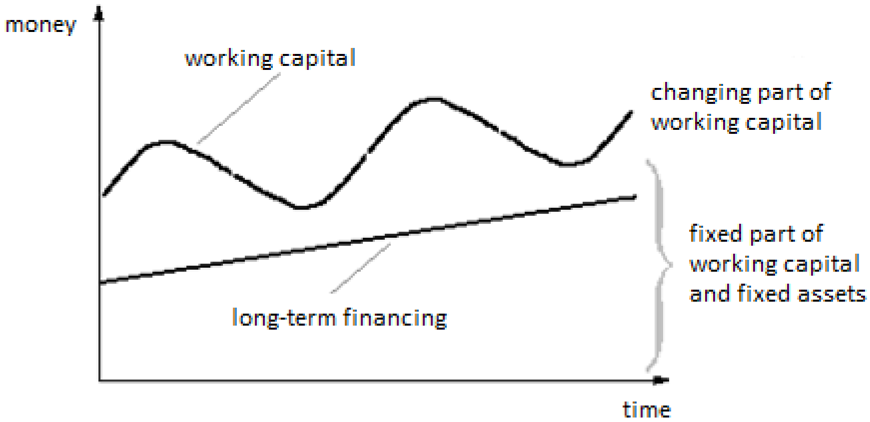
Company's property consists of total assets, i.e. fixed assets and current assets known also as liquid assets. We can see that property as fixed capital and liquid assets also. Generally liquid assets equal to current assets is defined as a sum of inventory, short term receivables (including all the accounts receivable for deliveries and services regardless of the maturity date) and short-term investments (cash and its equivalents) as well as short-term prepaid expenses [Mueller 1953; Graber 1948; Khoury 1999; Cote 1999]. Money tied in liquid assets serve enterprise as protection against risk [Merton 1999, p. 506; Lofthouse 2005; p. 27-28; Parrino 2008, p. 224-233, Poteshman 2005, s. 21-60] but that money also are considered as an investment. It is because the firm resigns from instant utilization of resources for future benefits [Levy 1999, p. 6; Reilly 1992, p. 6; Fabozzi 1999, p. 214]. In that paper the terms: *current assets* and *liquid assets* are treated as approximately equivalent and interchangeable [Michalski 2004].
Liquid assets level is the effect of processes linked to the production organization or services realization. So, it results from the processes that are operational by nature and therefore correspond to the willingness to produce on time products and services that are probably desired by customers [Baumol 1952, Beck 2005, Beranek 1963, Emery 1988, Gallinger 1986, Holmstrom 2001, Kim 1998, Kim 1978, Lyn 1996, Tobin 1958, Stone 1972, Miller 1966, Miller 1996, Myers 1998, Opler 1999, Rutkowski 2000]. It exerts influence mainly on the inventory level and belongs to the area of interest of operational management [Peterson 1979, s. 67-69; Orlicky 1975, s.17-19; Plossl 1985, s. 421-424]. Nevertheless, current assets are also the result of active customer winning and maintaining policy [Bougheas 2009]. Such policy is executed by finding an offer and a specific market where the product or service is sold. This policy consequences are reflected in the final products inventory level and accounts receivable in short term.
Among the motivating factors for investing in current assets, one may also mention uncertainty and risk. Due to uncertainty and risk, it is necessary to stock up circumspect (cautionary) cash, material and resources reserves that are inevitable in maintaining the continuity of production and producing final goods.
Many firms act in a fast changing environment where the prices of needed materials and resources are subject to constant change. Other factors – like exchange rates for instance, are very changeable, too. It justifies keeping additional cash sources allotted for realization of built-in call options (American type) by buying the raw materials more cheap than the long term expected equilibrium price would suggest.
Company's relationships with suppliers of materials, resources and services that are necessary to produce and sell final products usually result in adjourning the payments. Such situation creates Accounts payable and employees (who are to some extent internal services providers). Similarly, enterprise charged with obligatory payments will eventually face tax burdens. We will call both categories of obligations the non financial current obligations in order to differentiate between them and current obligations that result from taking on financial obligations, e.g. short term debt. Required payments postponement exerts impact on reducing the demand for these company's resources that are engaged in current asset financing. Current assets reduced by non financial current obligations (non financial short term obligations) are called net current assets. Net current assets are the resources invested by the company in current assets equated with the capital tied in these assets.

2 LIQUID ASSETS FINANCING STRATEGIES AND COST OF FINANCING

Net current assets (as a synonym for net liquid assets), i.e. current assets reduced by non financial current liabilities, are the sources tied by the firm during its realization of operational cycle. If it is required by the character of business, sources tied in liquid assets may be quite huge sums. This paper aims at analyzing the influence of investment in net liquid assets on enterprise value represented by a sum of future free cash flows discounted by the cost of financing the enterprise and next reflecting on the difference between investments in net

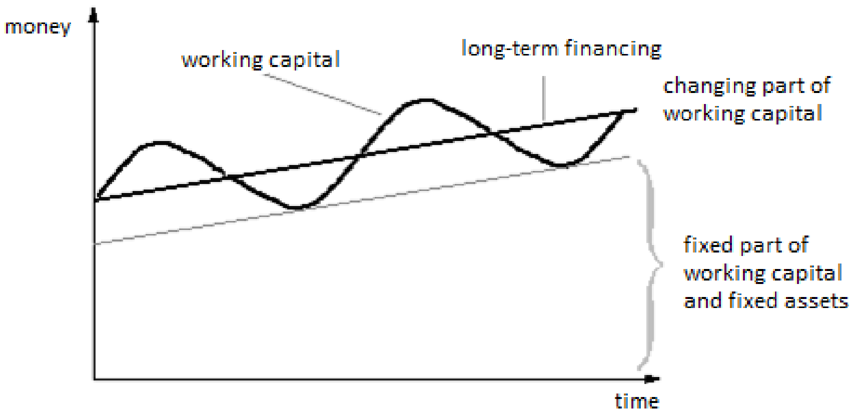
current assets and operational investments in fixed assets in terms of their effects on enterprise value growth.
Current assets investment strategies are the set of criteria and specific code of conduct revolved around attaining multiplication of owners wealth. Company's management implement such strategies into practice while making the crucial decisions concerning obtaining sources for financing current and future needs and defining ways and directions of utilization of these sources, taking into consideration at the same time: opportunities, limitations and business environment that are known to the board today. The same set of strategies come in consequence of market conditions and personal inclinations of the board members who are representatives of the owners (first of all – their attitude to risk). Based on this attitude, the board defines appropriate structure of current assets and financing sources. It is possible to apply one of the three liquid assets financing strategies (or their variations): aggressive, compromise or conservative.
Aggressive strategy consists in the significant part of the enterprise fixed demand and the whole enterprise variable demand on liquidity-linked financing sources coming from short term financing.

Figure 1 Aggressive liquid assets financing strategy



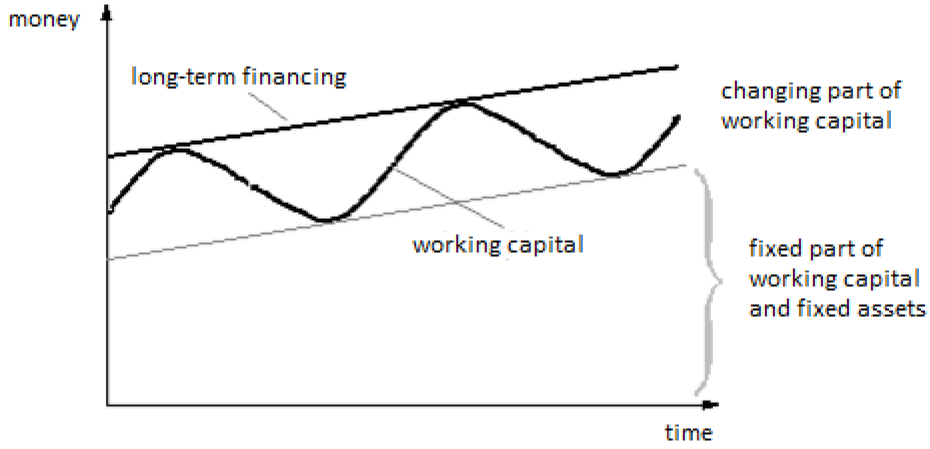
The Compromise version of liquid assets financing strategy aims at adjusting the needed financing period to the duration of period for which the enterprise needs these assets. As a result, the fixed share of current assets financing is based on long term capital. However, the variable share is financed by short term capital.

Figure 2 Compromise liquid assets financing strategy



The conservative liquid assets financing strategy leads to the situation where both the fixed and the variable level of current assets is maintained on the basis of long term financing.

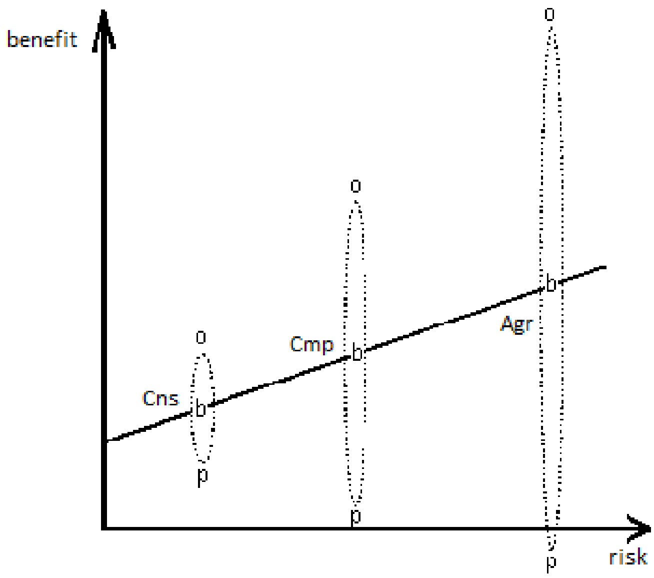
Figure 3 Conservative liquid assets financing strategy



3 LIQUID ASSETS FINANCING STRATEGY TO RISK RELATION

There is a relationship between the three above mentioned approaches based on the relation between expected benefit and risk (fig. 4). In case of capital providers for companies that have introduced this specific strategy it is usually linked with diversified claims to the rate of return from the amount of capital invested in the firm.

Figure 4 Diversified levels of expected benefits connected with different liquid assets financing strategies



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Where: Cns – conservative strategy, Cmp – Compromise strategy, Agr – aggressive strategy, b – base situation, o – situation better than expected, p – situation worse than expected.
Source: Hypothetical data.

The connection of these claims with the chosen way of financing may be insignificant (as it is shown on figure 5 or in variant 1 of the example beneath). Nevertheless, it also might be important to such a considerable degree that it will have an effect on the choice of strategy (figures 6 and 7).

Example. XYZ board of directors is pondering over the choice of current assets financing strategy. What is the best strategy provided that the aim of the management board is to minimize cost of financing liquid assets and maximize enterprise value?

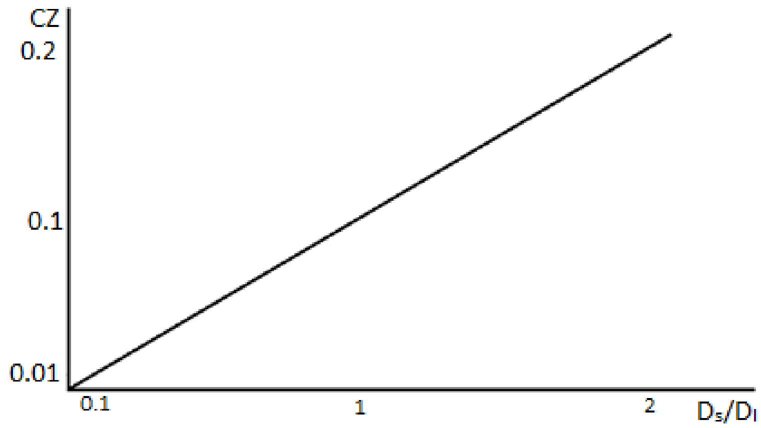
Equity/engaged capital ratio is 40% {E/(E+D) = 40%}. Anticipated annual sales revenues (CR) are 2000. Forecasted earnings before interest and taxes (EBIT) for XYZ will amount to about 50% of sales revenues (CR). Fixed assets (FA) will be going for around 1400, current assets (CA) will be constituting almost 30% of forecasted sales revenues (CR), property renewing will be close to its use (NCE = CAPEX), and changes in relations of net liquid assets constituents will be close to zero and might be omitted (ΔNWC = 0). The company may implement one of the three liquid assets financing strategies: the conservative one with such a relation of long run debt to short run debt that (D_s/D_l = 0,1), Compromise one ($D_s/(D_l)$ = 1) or the aggressive one ($D_s/(D_l)$ = 2). Accounts payable will be equal to 50% of current assets.

It is necessary to consider the influence of each strategy on the cost of enterprise financing capital rate and on enterprise value.

In the first variant, one must assume that capital providers seriously consider while defining their claims to rates of return the liquid assets financing strategy chosen by the company they invested in.

Let us also assume that the correction factor CZ function graph connected with strategy choice is even and linear (Figure 5).

Figure 5 The shape of correction factor CZ line as a function of D_s/D_l. Source: Hypothetical data.



CZ1 variant. We assume here that capital providers take into consideration the company's liquid assets financing strategy while defining their claims as regards the rates of return. Of course, aggressive strategy is perceived as more risky and therefore depending on investors risk aversion level, they tend to ascribe to the financed company applying aggressive strategy an additional expected risk premium. To put it simply, let us assume that ascribing the additional risk premium for applied liquid assets financing strategy is reflected in the value of *β* coefficient. For each strategy, the *β* coefficient will be corrected by the corrective coefficient CZ corresponding to that specific strategy in relation to the situation D_k/D_d = 0. XYZ risk premium will amount to 9%×(1+CZ) in relation of equity to foreign long term capital and 12%×(1+CZ) in relation of equity to short term debt level. Risk free rate is 4%, rate of return on market portfolio is 18%.

If Our company is a representative of A sector for which the non-leveraged risk coefficient β_u = 0.77. On the basis of Hamada relation, we can estimate the equity cost rate that is financing that enterprise in case of each of the three strategies in the first variant.

$$\beta_l = \beta_u \times \left(1 + (1-T) \times \frac{D}{E}\right) = 0.77 \times \left(1 + 0.81 \times \frac{0.4}{0.6}\right) = 1.19$$

Where: T – effective tax rate, D – enterprise financing capital coming from creditors (D_s+D_l), E – enterprise financing capital coming from owners, *β* – risk coefficient, β_u – risk coefficient linked with assets maintained by the firm (for an enterprise that has not applied the system of financing by creditors capital), β_l – risk coefficient for an enterprise that applying the system of financing by creditors capital (both the financial and operational risks are included).

For aggressive strategy (CZ = 0.2):

$$\beta_{i\,Agr}^{*} = \beta_u \times \left(1 + (1-T) \times \frac{D}{E}\right) \times (1 + CZ) = 0.77 \times \left(1 + 0.81 \times \frac{0.4}{0.6}\right) \times 1.2 = 1.19 \times 1.2 = 1.43$$

Where: CZ – risk premium correction factor dependent on the net liquid assets financing strategy

For compromise strategy (CZ = 0.1):

$$\beta_{i\,Cmp}^{*} = \beta_u \times \left(1 + (1-T) \times \frac{D}{E}\right) \times (1 + CZ) = 0.77 \times \left(1 + 0.81 \times \frac{0.4}{0.6}\right) \times 1.1 = 1.19 \times 1.1 = 1.31$$

For conservative strategy (CZ = 0.01):

$$\beta_{i\,cns}^{*} = \beta_u \times \left(1 + (1-T) \times \frac{D}{E}\right) \times (1 + CZ) = 0.77 \times \left(1 + 0.81 \times \frac{0.4}{0.6}\right) \times 1.01 = 1.19 \times 1.01 = 1.2$$

Thanks to that information, we can calculate cost of equity rates for every variant.

$$k_{e_{agr}} = \beta_i \times (k_m - k_{RF}) + k_{RF} = 1.43 \times 14\% + 4\% = 24\%$$

$$k_{e_{cmp}} = \beta_i \times (k_m - k_{RF}) + k_{RF} = 1.31 \times 14\% + 4\% = 22.3\%$$

$$k_{e_{cns}} = \beta_i \times (k_m - k_{RF}) + k_{RF} = 1.2 \times 14\% + 4\% = 20.8\%$$

Where: k – rate of return expected by capital donors and at the same time (from company's perspective) – enterprise cost of financing capital rate, k_e – for capital coming from owners (cost of equity rate), k_m – for average rate of return on typical investment on the market, k_{RF} – for risk free rate of return whose approximation is an average profitability of Treasury bills in the country where the investment is made.

Hence, since the risk premium for XYZ accounts for 9%×(1+CZ) in relation of equity to foreign long term capital, we can get long term debt cost rate:

$$k_{dl_{agr}} = k_{e_{agr}} - 9\% \times 1.2 = 24\% - 10.8\% = 13.2\%$$

$$k_{dl_{cmp}} = k_{e_{cmp}} - 9\% \times 1.1 = 22.3\% - 9.9\% = 12.4\%$$

$$k_{dl_{cns}} = k_{e_{cns}} - 9\% \times 1.01 = 20.8\% - 9.1\% = 11.7\%$$

Where: k_{dl} – for capital coming from long term creditors,

And consequently for short term:

$$k_{ds_{agr}} = k_{e_{agr}} - 12\% \times 1.2 = 24\% - 14.4\% = 9.6\%$$

$$k_{ds_{cmp}} = k_{e_{cmp}} - 12\% \times 1.1 = 22.3\% - 13.2\% = 9.1\%$$

$$k_{ds_{cns}} = k_{e_{cmp}} - 12\% \times 1.01 = 20.8\% - 12.1\% = 8.7\%$$

Where: k_{ds} – for capital coming from short term creditors,

As a result, cost of capital rate will amount to:

$$CC = \frac{E}{E + D_l + D_s} \times k_e + \frac{D_l}{E + D_l + D_s} \times k_{dl} \times (1-T) + \frac{D_s}{E + D_l + D_s} \times k_{ds} \times (1-T)$$

However, for each strategy, this cost rate will be on another level (calculations in the table below).

Table 1 Cost of capital and changes in enterprise value depending on the choice of strategy. Source: Hypothetical data

	Aggressive	*Compromise*	*Conservative*
Sales revenues (CR)	2000	2000	2000
Fixed assets (FA)	1400	1400	1400
Current assets (CA)	600	600	600
Total assets (TA) = Total liabilities (TL)	2000	2000	2000
(AP)	300	300	300
Engaged capital (E+D)	1700	1700	1700
Equity (E)	680	680	680
Long term debt (D_l)	340	510	927

Short term debt (D_s)	680	510	93
Earnings before interest and taxes (EBIT)	1000	1000	1000
Net operational profit after taxation (NOPAT)	810	810	810
Free cash flows from 1 to n period ($FCF_{1..n}$)	810	810	810
Free cash flows in 0 (FCF_o)	-1700	-1700	-1700
Risk premium correction factor CZ	0.2	0.1	0.01
Risk coefficient β_l	1.428	1.309	1.2019
Equity cost (k_e)	24%	22.3%	20.8%
Cost of long term debt (k_{dl})	13.2%	12.4%	11.7%
Cost of short term debt (k_{ds})	9.6%	9.1%	8.7%
Cost of capital financing enterprise (CC)	14.8%	14.2%	**13.9%**
Enterprise value growth (ΔV)	3758	4017	**4127**

As it is shown in the table, cost of enterprise financing capital rates are different for different approaches to liquid assets financing. The lowest rate is observed in conservative strategy.

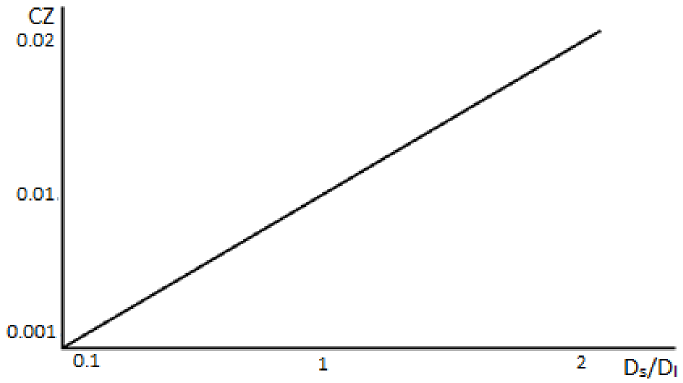
$$CC_{cns} = \frac{680}{1700} \times 20.8\% + \frac{340}{1700} \times 11.7\% \times (1-0.19) + \frac{680}{1700} \times 8.7\% \times (1-0.19) = 13.9\%$$

What results in the highest expected growth of enterprises value:

$$\Delta V_{cns} = FCF_0 + \frac{FCF_{1...n}}{CC} = -1700 + \frac{810}{0.139} = 4127$$

In the CZ2 variant, we will also assume that capital providers while defining their claims to rates of return take into consideration the company's liquid assets financing strategy to a lesser extent. Obviously, the aggressive strategy is perceived as more risky and therefore, depending on their risk aversion, they tend to ascribe an additional risk premium for an enterprise that implemented this type of strategy.

Figure 6 Correction line depending on the D_s/D_l relation in the second variant. Source: Hypothetical data.



For conservative strategy, XYZ risk premium is equal to 9%×(1+CZ) in relation of equity to long term debt and 12%×(1+CZ) in relation of equity to short term debt. Risk free rate of return is 4%, rate of return on market portfolio is 18%.
Our company is a representative of a sector for which non-leveraged risk coefficient $\beta_u = 0.77$.
On the basis of Hamada relation, we may estimate the cost rate of equity financing this enterprise in case of each of the three strategies.
We are given all necessary information to assess cost of enterprise financing capital rate for the firm applying the given type of liquid assets financing strategy.
For each strategy the cost rate *CC* will be on another level (calculations in the table below).

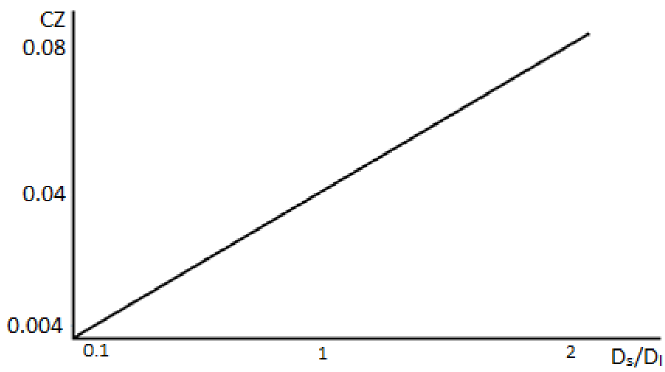
Table 2 Cost of capital and changes in enterprise value depending on the choice of strategy in variant CZ2; Source: Hypothetical data.

	Aggressive	*Compromise*	*Conservative*
Sales revenues (CR)	2000	2000	2000
Fixed assets (FA)	1400	1400	1400
Current assets (CA)	600	600	600
Total assets (TA) = Total liabilities (TL)	2000	2000	2000
Accounts payable (AP)	300	300	300
Engaged capital (E+D)	1700	1700	1700
Equity (E)	680	680	680
Long term debt (D_l)	340	510	927
Short term debt (D_s)	680	510	93
Earnings before interest and taxes (EBIT)	1000	1000	1000
Net operational profit after taxation (NOPAT)	810	810	810
Free cash flows from 1 to n ($FCF_{1..n}$)	810	810	810
Free cash flows in 0 (FCF_o)	-1700	-1700	-1700
Risk premium correction CZ	0.02	0.01	0.001
Risk coefficient β_l	1.2138	1.2019	1.19119
Equity cost (k_e)	21%	20.8%	20.7%
Long term debt cost (k_{dl})	11.8%	11.7%	11.7%
Short term debt cost (k_{ds})	8.8%	8.7%	8.7%
Capital cost of capital financing the enterprise (CC)	**13.15%**	13.30%	13.81%
Enterprise value growth (ΔV)	**4461**	4391	4166

As it is shown in table 2, taking into consideration the risk premium resulting from implementation of a certain liquid assets financing strategy has an additional impact on the enterprise financing capital. Enterprise financing capital cost rates are different for different approaches to liquid assets financing. In this variant, the lowest level is observed in aggressive strategy. As a consequence, the highest enterprise value growth is characteristic for this type of strategy.

In the third CZ3 variant, we also assume that capital providers to a lesser extent consider while defining their claims to rates of return the liquid assets financing strategy chosen by the company they invested in.

Figure 7 Correction line depending on the D_k/D_d relation in the CZ3 variant; Source: Hypothetical data.



For conservative strategy, XYZ risk premium amounts to 9%×(1+CZ) in relation of equity to long term debt level and 12%×(1+CZ) in relation of equity to short term debt. Risk free rate is 4%, rate of return on market portfolio is 18%.

Our company is a representative of sector W for which non-leveraged risk coefficient $\beta_u = 0.77$. On the basis of Hamada relation we may estimate enterprise financing equity cost rate in case of each of the three strategies.
We have all necessary information to assess the enterprise financing capital cost for the firm applying the given type of liquid assets financing strategy. For each strategy, capital cost rate will be on another level (calculations in Table 3).

Table 3 Cost of capital and changes in enterprise value depending on the choice of strategy in the CZ3 variant; Source: Hypothetical data

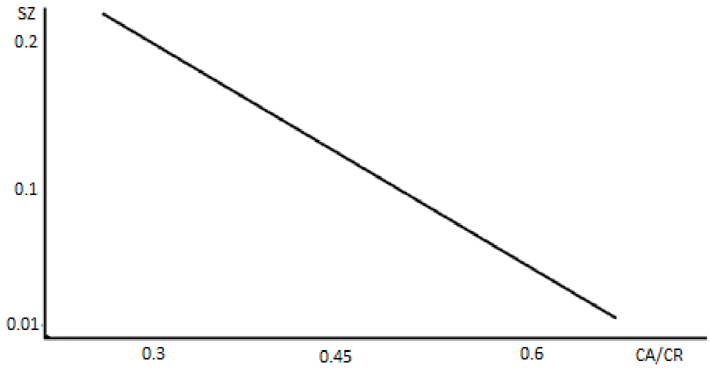
	Agressive	*Compromise*	*Consevative*
Sales revenues (CR)	2000	2000	2000
Fixed assets (FA)	1400	1400	1400
Current assets (CA)	600	600	600
Total assets (TA) = Total liabilities (TL)	2000	2000	2000
Accounts payable (AP)	300	300	300
Engaged capital (E+D)	1700	1700	1700
Equity (E)	680	680	680
Long term debt (D_l)	340	510	927
Short term debt (D_s)	680	510	93
Earnings before interest and taxes (EBIT)	1000	1000	1000
Net operational profit after taxation (NOPAT)	810	810	810
Free cash flows from 1 to n ($FCF_{1..n}$)	810	810	810
Free cash flows from 0 (FCF_o)	-1700	-1700	-1700
Risk premium correction CZ	0.08	0.04	0.004
Risk coefficient β_l	1.2852	1.2376	1.19476
Equity cost (k_e)	22%	21.3%	20.7%
Lon term debt cost (k_{dl})	12.3%	12%	11.7%
Short term debt cost (k_{ds})	9%	8.9%	8.7%
Enterprise financing capital cost (CC)	13.7%	**13.6%**	13.8%
Enterprise value growth (ΔV)	4207	**4261**	4153

As it is shown in table 3, taking into consideration the risk premium resulting from implementation of a certain liquid assets financing strategy has an additional impact on the enterprise financing capital. Enterprise financing capital cost rates are different for different approaches to liquid assets financing. In this variant, the lowest level is observed in aggressive strategy. As a consequence, the highest enterprise value growth is characteristic for this type of strategy.

4 LIQUID ASSETS INVESTMENT STRATEGIES AND COST OF FINANCING

Next it is necessary to consider the influence of each strategy of investment in the liquid assets on the rate of cost of capital financing enterprise and that influence on the enterprise value.
In the first variant, one must assume that capital providers seriously consider while defining their claims to rates of return the liquid assets investment strategy chosen by the company they invested in.
Let us also assume that the correction SZ function graph connected with strategy choice could be even and linear (figure 8).

Figure 8 The shape of line of correction SZ as a function of CA/CR in the SZ1 variant; Source: Hypothetical data.



SZ1 variant. We assume here that capital providers take into consideration the company's liquid assets investment strategy while defining their claims as regards the rates of return. Of course, **restrictive** strategy is perceived as more risky and therefore depending on investors risk aversion level, they tend to ascribe to the financed company applying restrictive strategy an additional expected risk premium. To put it simply, let us assume that ascribing the additional risk premium for applied liquid assets investment strategy is reflected in the value of β risk coefficient. For each strategy, the β risk coefficient will be corrected by the corrective coefficient SZ corresponding to that specific strategy in relation to the CA/CR situation.

The risk free rate is 4%, and rate of return on market portfolio is 18%. If Our company is a representative of A sector for which the non-leveraged risk coefficient β_u = 0.77. On the basis of Hamada relation, we can estimate the equity cost rate that is financing that enterprise in case of each of the three strategies in the SZ1 variant.

$$\beta_l = \beta_u \times \left(1 + (1 - T) \times \frac{D}{E}\right) = 0.77 \times \left(1 + 0.81 \times \frac{0.4}{0.6}\right) = 1.19$$

Where: T – effective tax rate, D – enterprise financing capital coming from creditors (a sum of short term debt and long term debt $D=D_s+D_l$), E – enterprise financing capital coming from owners of the firm, β – risk coefficient, β_u – risk coefficient for an assets of the enterprise that not use debt, β_l – risk coefficient for an enterprise that applying the system of financing by creditors capital (here we have both asset and financial risk).

For restrictive strategy, where CA/CR is 0.3; the SZ risk premium is 0.2:

$$\beta^*_{l\,r} = \beta_u \times \left(1 + (1 - T) \times \frac{D}{E}\right) \times (1 + SZ) = 0.77 \times \left(1 + 0.81 \times \frac{0.4}{0.6}\right) \times 1.2 = 1.19 \times 1.2 = 1.43$$

Where: SZ – risk premium correction dependent on the liquid assets investment strategy.

For moderate strategy, where CA/CR is 0.45 the SZ risk premium is 0.1:

$$\beta^*_{l\,m} = \beta_u \times \left(1 + (1 - T) \times \frac{D}{E}\right) \times (1 + SZ) = 0.77 \times \left(1 + 0.81 \times \frac{0.4}{0.6}\right) \times 1.1 = 1.19 \times 1.1 = 1.31$$

For flexible strategy, where CA/CR is 0.6 the SZ risk premium is 0.01:

$$\beta^*_{l\,f} - \beta_u \times \left(1 + (1 - T) \times \frac{D}{E}\right) \times (1 + SZ) - 0.77 \times \left(1 + 0.01 \times \frac{0.4}{0.6}\right) \times 1.01 - 1.19 \times 1.01 - 1.2$$

Using that information we can calculate cost of equity rates for each liquid assets investment strategy. For restrictive strategy:

$$k_{e_r} = \beta_l \times (k_m - k_{RF}) + k_{RF} = 1.43 \times 14\% + 4\% = 24\%;$$

For moderate strategy:

$$k_{e_m} = \beta_l \times (k_m - k_{RF}) + k_{RF} = 1.31 \times 14\% + 4\% = 22.3\%;$$

And for flexible strategy:

$$k_{e_f} = \beta_l \times (k_m - k_{RF}) + k_{RF} = 1.2 \times 14\% + 4\% = 20.8\%.$$

Where: k – rate of return expected by capital donors and at the same time (from company's perspective) – enterprise cost of financing capital rate, k_e – for cost rate of the equity, k_{dl} – for long term debt rate, k_{ds} – for short term debt rate, k_m – for average rate of return on typical investment on the market, k_{RF} – for risk free rate of return whose approximation is an average profitability of treasury bills in the country where the investment is made.

In similar way, we can calculate the risk premiums for XYZ alternative rates. We know that long term debt rates differ for 9%×(1+SZ) in relation of equity to long term debt. From that we can get long term debt cost rates for each alternative strategy. For restrictive strategy:

$k_{dl_r} = k_{e_r} - 9\% \times 1.2 = 24\% - 10.8\% = 13.2\%;$

For moderate strategy:

$k_{dl_m} = k_{e_m} - 9\% \times 1.1 = 22.3\% - 9.9\% = 12.4\%;$

And for flexible strategy:

$k_{dl_f} - k_{e_f} - 9\% \times 1.01 - 20.8\% - 9.1\% - 11.7\%.$

Next we can calculate the risk premiums for XYZ alternative cost of short term rates. We know that short term debt rates differ for 12%×(1+SZ) in relation of cost of equity rates to short term debt rates. From that we can get short term debt cost rates for each alternative strategy. For restrictive strategy:

$k_{ds_r} = k_{e_r} - 12\% \times 1.2 = 24\% - 14.4\% = 9.6\%;$

For moderate strategy:

$k_{ds_m} = k_{e_m} - 12\% \times 1.1 = 22.3\% - 13.2\% = 9.1\%;$

And for flexible strategy:

$k_{ds_f} = k_{e_f} - 12\% \times 1.01 = 20.8\% - 12.1\% = 8.7\%.$

As a result, cost of capital rate will amount to:

$$CC = \frac{E}{E + D_l + D_s} \times k_e + \frac{D_l}{E + D_l + D_s} \times k_{dl} \times (1 - T) + \frac{D_s}{E + D_l + D_s} \times k_{ds} \times (1 - T)$$

However, for each strategy – this cost rate will be on another level (calculations in the table 4. below).

Table 4 Cost of capital and changes in enterprise value depending on the choice of liquid assets investment strategy; Source: Hypothetical data

Liquid assets investment strategy	*Restrictive*	*Moderate*	*Flexible*
Cash Revenues (CR)	2000	2080	2142,4
Fixed assets (FA)	1400	1445	1480
Current assets (CA)	600	936	1285
Total assets (TA) = Total liabilities (TL)	2000	2381	2765
Accounts payable (AP)	300	468	643
Capital invested ($E+D_l+D_s$)	1700	1913	2122
Equity (E)	680	765	849
Long-term debt (D_l)	340	383	424
Short-term debt (D_s)	680	765	849
EBIT share in CR	0.5	0.45	0.40
Earnings before interests and taxes (EBIT)	1000	936	857
Net operating profit after taxes (NOPAT)	810	758	694
Free Cash Flows in 1 to n periods ($FCF_{1..n}$)	810	758	694
Initial Free Cash Flows in year 0 (FCF_o)	-1700	-1913	-2122
SZ risk Premium correction	0.2	0.1	0.01
Leveraged and corrected risk coefficient β_l	1.428	1.309	1.2019
Cost of equity rate (k_e)	24%	22.3%	20.8%
Long-term debt rate (k_{dl})	13.2%	12.4%	11.7%
Short-term debt rate (k_{ds})	9.6%	9.1%	8.7%
Cost of capital (CC)	14.8%	13.9%	**13.1%**

Firm value growth (ΔV)	**3758**	3542	3198

As it is shown in the table, rates of the cost of capital financing the firm are different for different approaches to liquid assets investment. The lowest rate: CC = 13.1%; is observed in flexible strategy because that strategy is linked with the smallest level of risk but the highest firm value growth is linked with restrictive strategy of investment in net liquid assets.

Cost of capital for restrictive strategy of investment in working capital:

$$CC_r = \frac{680}{1700} \times 24\% + \frac{340}{1700} \times 13.2\% \times (1 - 0.19) + \frac{680}{1700} \times 9.6\% \times (1 - 0.19) = 14.8\%$$

Expected growth of enterprise value for that strategy:

$$\Delta V_r = FCF_0 + \frac{FCF_{1...n}}{CC} = -1700 + \frac{810}{0.148} = 3758.$$

Cost of capital for moderate strategy of investment in working capital:

$$CC_m = \frac{765}{1913} \times 22.3\% + \frac{383}{1913} \times 12.4\% \times (1 - 0.19) + \frac{765}{1913} \times 9.1\% \times (1 - 0.19) = 13.9\%;$$

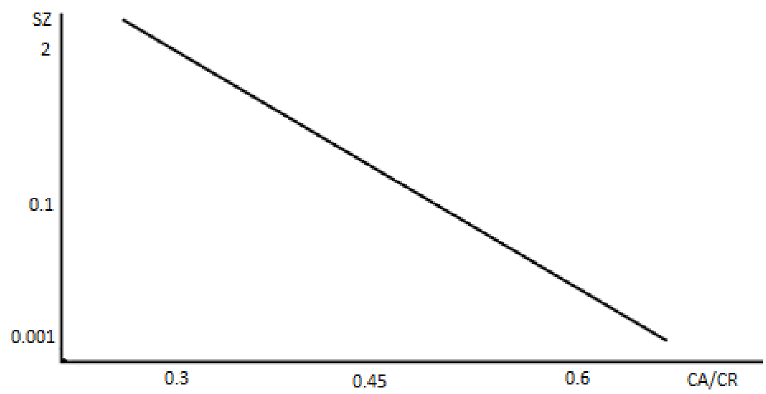
Expected growth of enterprise value for that strategy: 3542

Cost of capital for flexible strategy of investment in working capital: 13.1%

Expected growth of enterprise value for flexible strategy: 3196.

In the next, SZ2 variant, we will also assume that capital providers while defining their claims to rates of return take into consideration the company's net working investment strategy to a lesser extent. Obviously, the restrictive strategy is perceived as more risky than moderate and flexible. Depending on their risk aversion, they tend to ascribe an additional risk premium for an enterprise that implemented this type of strategy. As presented on figure 9, investors in SZ2 variant, have stronger risk aversion than in SZ1 situation.

Figure 9 The shape of line of correction SZ as a function of CA/CR in the SZ2 variant. Source: Hypothetical data.



In the table 5. There are calculations for variant SZ2. For each strategy the cost of capital rate *CC* will be on another level.

Table 5 Cost of capital and changes in enterprise value depending on the choice of strategy of investment in liquid assets in variant SZ2. Source: Hypothetical data

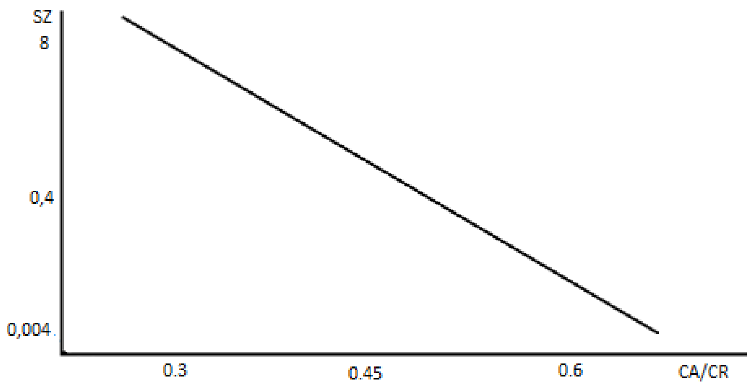
Liquid assets investment strategy	*Restrictive*	*Moderate*	*Flexible*
Cash Revenues (CR)	2000	2080	2142
Fixed assets (FA)	1400	1445	1480
Current assets (CA)	600	936	1285
Total assets (TA) = Total liabilities (TL)	2000	2381	2765
Accounts payable (AP)	300	468	643
Capital invested ($E+D_l+D_s$)	1700	1913	2122
Equity (E)	680	765	849

Long-term debt (D_l)	340	383	424
Short-term debt (D_s)	680	765	849
EBIT share in CR	0.5	0.45	0.4
Earnings before interests and taxes (EBIT)	1000	936	857
Net operating profit after taxes (NOPAT)	810	758	694
Free Cash Flows in 1 to n periods ($FCF_{1..n}$)	810	758	694
Initial Free Cash Flows in year 0 (FCF_o)	-1700	-1913	-2122
SZ risk Premium correction	2	0.1	0.001
Leveraged and corrected risk coefficient β_l	3.57	1.31	1.19
Cost of equity rate (k_e)	54%	22.3%	20.7%
Long-term debt rate (k_{dl})	27%	12.4%	11. 7%
Short-term debt rate (k_{ds})	18%	9.1%	8.7%
Cost of capital (CC)	31.8%	13.9%	**13%**
Firm value growth (ΔV)	848	**3542**	3230

As it is shown in table 5, taking into consideration the risk premium resulting from implementation of a certain liquid assets strategy has an additional impact on the enterprise financing capital and its rate. Enterprise financing capital cost rates are different for different approaches to liquid assets investment. In this variant SZ2, similarly as to the variant SZ1 presented in table 4., the lowest level of cost of capital is observed in flexible strategy. But, the highest enterprise value growth is characteristic for moderate strategy.

In the third, SZ3 variant. The restrictive and moderate strategies are more risky than flexible. Depending on their risk aversion, they tend to ascribe an additional risk premium for an enterprise that implemented this type of strategy. As presented on figure 10, investors in SZ3 variant, have stronger risk aversion than in SZ1 and SZ2 situations.

Figure 10 The shape of line of correction SZ as a function of CA/CR in the SZ3 variant. Source: Hypothetical data.



In the table 6 there are calculations for variant SZ3. For each strategy the cost of capital rate *CC* will be on another level.

Table 6 Cost of capital and changes in enterprise value depending on the choice of strategy of investment in liquid assets in the SZ3 variant. Source: Hypothetical data.

Liquid assets investment strategy	*Restrictive*	*Moderate*	*Flexible*
Cash Revenues (CR)	2000	2080	2142
Fixed assets (FA)	1400	1445	1480
Current assets (CA)	600	936	1285
Total assets (TA) = Total liabilities (TL)	2000	2381	2765
Accounts payable (AP)	300	468	643
Capital invested ($E+D_l+D_s$)	1700	1913	2122
Equity (E)	680	765	849

Long-term debt (D_l)	340	383	424
Short-term debt (D_s)	680	765	849
EBIT share in CR	0.5	0.45	0.4
Earnings before interests and taxes (EBIT)	1000	936	857
Net operating profit after taxes (NOPAT)	810	758	694
Free Cash Flows in 1 to n periods ($FCF_{1..n}$)	810	758	694
Initial Free Cash Flows in year 0 (FCF_o)	-1700	-1913	-2122
SZ risk Premium correction	8	0.4	0.004
Leveraged and corrected risk coefficient β_l	10.7	1.7	1.2
Cost of equity rate (k_e)	154%	27.3%	20.7%
Long-term debt rate (k_{dl})	73%	14.7%	11.7%
Short-term debt rate (k_{ds})	46%	10.5%	8.7%
Cost of capital (CC)	88%	16.7%	13%
Firm value growth (ΔV)	-782	2620	3219

As it is shown in table 6, taking into consideration the risk premium resulting from implementation of a certain liquid assets investment strategy has an additional impact on the cost of capital. Enterprise financing capital cost rates are different for different approaches to liquid assets investment strategy. In this SZ3 variant, the lowest level of the cost of capital is observed in flexible strategy. But as a consequence, the highest enterprise value growth is characteristic also for this type of strategy, what is differ to results from variants SZ1 and SZ2. Here we have the highest level of risk aversion and as consequence the firm management wanting to maximize the firm value need to prefer more safe solution like flexible strategy.

LIQUID ASSETS INVESTMENT-FINANCING STRATEGIES AND COST OF FINANCING

Last part of our consideration is influence of each liquid assets strategy both from investment and financing perspective and their influence on cost of financing and that influence on the enterprise value.

SZCZ1 variant. In the first SZCZ1 variant, capital suppliers risk aversion is on the smallest level. That situation is presented in table 7.

Table 7 Cost of capital and changes in enterprise value depending on the choice of liquid assets investment and financing strategies. Source: Hypothetical data

Liquid assets investment and financing strategy	*Restrictive-Aggresive*	*Restrictive-Conservative*	*Flexible-Aggressive*	*Flexible-Conservative*
Cash Revenues (CR)	2000	2000	2142	2142
Fixed assets (FA)	1400	1400	1480	1480
Current assets (CA)	600	600	1285	1285
Total assets (TA) = Total liabilities (TL)	2000	2000	2765	2765
Accounts payable (AP)	300	300	643	643
Capital invested ($E+D_l+D_s$)	1700	1700	2123	2123
Equity (E)	680	680	849	849
Long-term debt (D_l)	340	927	425	1158
Short-term debt (D_s)	680	93	849	116
EBIT share in CR	0.5	0.5	0.40	0.40
Earnings before interests and taxes (EBIT)	1000	1000	857	857
Net operating profit after taxes (NOPAT)	810	810	694	694
Free Cash Flows in 1 to n periods ($FCF_{1..n}$)	810	810	694	694
Initial Free Cash Flows in year 0	-1700	-1700	-2123	-2123

(FCF_o)				
CZ+SZ risk Premium correction	0.283	0.2	0.2	0.014
Leveraged and corrected risk coefficient β_l	1.53	1.43	1.43	1.21
Cost of equity rate (k_e)	25.4%	24%	24%	20.9%
Long-term debt rate (k_{dl})	13.8%	13.2%	13.2%	11.8%
Short-term debt rate (k_{ds})	10%	9.6%	9.6%	8.7%
Cost of capital (CC)	15.6%	15.9%	14.8%	**14%**
Firm value growth (ΔV)	**3485**	3410	2554	2856

As it is shown in the table 7, rates of the cost of capital financing the firm are different for different approaches to liquid assets investment. The lowest rate: CC = 14%; is observed in flexible-conservative strategy because that strategy is linked with the smallest level of risk but the highest firm value growth is linked with restrictive-aggressive strategy because in variant CZSZ1 we have the firm with the smallest level of risk aversion.

In the next, CZSZ2 variant, capital suppliers risk aversion is on the moderate level. That situation is presented in table 8.

Table 8 Cost of capital and changes in enterprise value depending on the choice of liquid assets investment and financing strategies. Source: Hypothetical data.

Liquid assets investment and financing strategy	*Restrictive-Aggresive*	*Restrictive-Conservative*	*Flexible-Aggressive*	*Flexible-Conservative*
Cash Revenues (CR)	2000	2000	2142	2142
Fixed assets (FA)	1400	1400	1480	1480
Current assets (CA)	600	600	1285	1285
Total assets (TA) = Total liabilities (TL)	2000	2000	2765	2765
Accounts payable (AP)	300	300	643	643
Capital invested $(E+D_l+D_s)$	1700	1700	2123	2123
Equity (E)	680	680	849	849
Long-term debt (D_l)	340	927	425	1158
Short-term debt (D_s)	680	93	849	116
EBIT share in CR	0.5	0.5	0.40	0.40
Earnings before interests and taxes (EBIT)	1000	1000	857	857
Net operating profit after taxes (NOPAT)	810	810	694	694
Free Cash Flows in 1 to n periods $(FCF_{1..n})$	810	810	694	694
Initial Free Cash Flows in year 0 (FCF_o)	-1700	-1700	-2123	-2123
CZ+SZ risk premium correction	2.0	2.0	0.02	0.0014
Leveraged and corrected risk coefficient β_l	3.57	3.57	1.21	1.19
Cost of equity rate (k_e)	54%	54%	21%	20.7%
Long-term debt rate (k_{dl})	27%	27%	11.8%	11.7%
Short-term debt rate (k_{ds})	18%	18%	8.8%	8.7%
Cost of capital (CC)	31.8%	34.3%	**13.2%**	13.8%
Firm value growth (ΔV)	848	661	**3157**	2903

As it is shown in the table 8, rates of the cost of capital financing the firm are different for different approaches to liquid assets investment. The lowest rate: CC = 13.2%; is observed in flexible-aggressive strategy because that strategy is linked with the smallest level of risk and highest level of cheaper short term debt also the highest firm value growth is linked with flexible-aggressive strategy because in variant CZSZ2 we have the firm with the moderate level of risk aversion so previously noted as better restrictive-aggressive is here too risky.

In the third, CZSZ3 variant. In the first SZCZ1 variant, capital suppliers risk aversion is on the smallest level. That situation is presented in table 9.

Table 9 Cost of capital and changes in enterprise value depending on the choice of liquid assets investment and financing strategies. Source: Hypothetical data.

Liquid assets investment and financing strategy	*Restrictive -Aggresive*	*Restrictive-Conservative*	*Flexible-Aggressive*	*Flexible-ConservativeD*
Cash Revenues (CR)	2000	2000	2143	2143
Fixed assets (FA)	1400	1400	1480	1480
Current assets (CA)	600	600	1285	1285
Total assets (TA) = Total liabilities (TL)	2000	2000	2765	2765
Accounts payable (AP)	300	300	643	643
Capital invested ($E+D_l+D_s$)	1700	1700	2123	2123
Equity (E)	680	680	849	849
Long-term debt (D_l)	340	927	425	1158
Short-term debt (D_s)	680	93	849	116
EBIT share in CR	0.5	0.5	0.40	0.40
Earnings before interests and taxes (EBIT)	1000	1000	857	857
Net operating profit after taxes (NOPAT)	810	810	694	694
Free Cash Flows in 1 to n periods ($FCF_{1..n}$)	810	810	694	694
Initial Free Cash Flows in year 0 (FCF_o)	-1700	-1700	-2123	-2123
SZ risk Premium correction	8.0	8.0	0.08	0.0057
Leveraged and corrected risk coefficient β_l	10.71	10.71	1.29	1.2
Cost of equity rate (k_e)	154%	154%	22%	20.8 %
Long-term debt rate (k_{dl})	73%	73%	12.3%	11.7%
Short-term debt rate (k_{ds})	46%	46%	9%	8.7%
Cost of capital (CC)	88%	96%	**13.7%**	13.9%
Firm value growth (ΔV)	-783	-855	**2940**	2887

As it is shown in the table 9, rates of the cost of capital financing the firm are different for different approaches to liquid assets investment. The lowest rate: CC = 13.7%; is observed in flexible-aggressive strategy because that strategy is linked with the smallest level of risk and highest level of cheaper short term debt also the highest firm value growth is linked with flexible-aggressive strategy because in variant CZSZ3 we have the firm with the moderate level of risk aversion so previously noted as better restrictive-aggressive is here too risky.

5 POLISH EMPIRICAL DATA

Polish risk free rate in 2009 was 4.84% and in 2010 was 3.7% [MONEY]. The difference between rate of return on market portfolio and risk free rate in 2009 was 7.66% and for 2010 that difference was 8% [DAMODARAN]. Unleveraged betas for total market were 0.82 in 2009 and 0.83 in 2010. Higher risk shown by beta coefficient, should results with more flexible and more conservative liquid assets strategies. Average cost of debt rate in 2009 was 9.00% and for 2010 it was 8.5%.

Table 10 Liquid assets relation to key indicators in Polish firms in 2009. Source: own calculations.

2009	*Revenues*	*EBIT*	*Assets*	*Current Assets*	*CA / Rev*	*CA / Assets*	*CA / EBIT*	*Current Ratio*	*Quick Ratio*	*Cash Ratio*
Size of population	2856	2856	2856	2856	2856.00	2856.00	2856.00	2856.00	2856.00	2856.00
Arithmetic mean	66 176 940	3 289 922	49532029	22404388	0.34	0.45	6.81	7.21	5.17	1.86
Standard deviation	90 381 963	8 230 746	71768285	27919740	-	-	-	111.03	81.85	18.91
Median	37 859 931	1 534 276	26339963	13525355	0.36	0.51	8.82	1.75	1.27	0.26
winsorized mean	45 114 168	2 082 973	32775603	15858126	0.35	0.48	7.61	2.03	1.47	0.43
Truncated mean	130 015 140	7 478 086	95729771	46272575	0.36	0.48	6.19	5.45	4.16	1.78
Skewness	5	3	5	5	-	-	-	39.18	42.29	27.48
Maximum	1 639 519 924	101 837 720	988862988	603391000	-	-	-	5207.60	5207.60	5207.60
Minimum	-104 061 878	-84 914 093	489	315	-	-	-	0.00	-5.21	0.00

Table 11 Capital rates as measures of general risk in Polish firms in 2009. Source: own calculations. Source: own calculations

2009	***Equity***	***Interests***	***Long-term debt (Dl)***	***Short-term debt (Ds)***	***kd***	***ke***	***CC***
Size of population	2856	2856	2856	2856	2856	2856	2856
Arithmetic mean	29 178 361	1113581.622	5033164.568	12876188.92	6.22%	14.24%	10.74%
Standard deviation	120 172 598	3420447.481	19433709.95	16044456.23	-	-	-
Median	13 314 674	242 607	291 454	6 745 217	3.45%	13.81%	10.00%
winsorized mean	**16 514 152**	**388 204**	**1 389 657**	**8 658 690**	3.86%	14.22%	10.02%
Truncated mean	51 257 743	1 700 171	5 918 404	27 618 444	5.07%	14.45%	10.36%
Skewness	**42**	**6**	**12**	**2**	-	-	-
Maximum	5 925 924 989	51 474 502	512 475 562	99 940 959	-	-	-
Minimum	**- 158 708 728**	**- 46 179 700**	-	**20**	-	-	-

According to the model discussed in previous part of the paper, the liquidity strategies changes should be connected with general level of risk in Polish firms situation.

Figure 11 The expected change in liquid assets indicators after changes in risk indicators. Source: own study.

Change	Indicator	Change
↗	β	↘
↗	CC	↘
↗↘	CA/Rev	↘↗
↗↘	CA/Assets	↘↗
↗↘	CA/EBIT	↘↗

After the risk indicator β go up (at Figure 11 the arrow in the first left column), at least two sources of change are influenced in firms. First, the higher cost of capital make the investment in liquid assets more costly, so it works up to make liquid assets levels smaller. In the same time, the higher risk in general, cause the managing team of the firms to think more conservative and more flexible about the liquidity levels. It is a part of their risk aversion feelings about general situation in the firm. That is illustrated by the couple of arrows in different destinations (the first up, and the second down) but it is not true that both influences are the same, almost always the one of them is stronger than the other.

Table 12 Liquid assets relation to key indicators in Polish firms in 2010. Source: own calculations.

2010	*Revenues*	*EBIT*	*Assets*	*Current Assets*	*CA / Rev*	*CA / Assets*	*CA / EBIT*	*Current Ratio*	*Quick Ratio*	*Cash Ratio*
Size of population	2903	2903	2903	2903	2903.00	2903.00	2903.00	2903.00	2903.00	2903.00
Arithmetic mean	75 266 902	6 135 640	51663050	53767162	0.71	1.04	8.76	59.14	30.76	3.97
Standard deviation	278 975 775	113 898 012	72725326	1510390880	-	-	-	2806.80	1349.64	120.80
Median	39 484 482	1 560 240	28389218	14476417	0.37	0.51	9.28	1.76	1.27	0.25
winsorized mean	46 824 684	2 111 314	34500890	17000175	0.36	0.49	8.05	2.02	1.47	0.41
Truncated mean	134 441 882	7 801 121	100574432	48727613	0.36	0.48	6.25	5.29	4.13	1.78

Skewness	41	51	5	53	-	-	-	53.80	53.64	51.71
Maximum	13 680 575 399	6 010 543 074	886209000	81171586021	-	-	-	151164.19	151164.19	151164.19
Minimum	-7 629 000	-168 002 474	5717	-18921047	-	-	-	-2.18	-3.31	0.00

Table 13 Capital rates as measures of general risk in Polish firms in 2010. Source: own calculations.

2010	*Equity*	*Interests*	*Long-term debt (Dl)*	*Short-term debt (Ds)*	*kd*	*ke*	*CC*
Size of population	2903	2903	2903	2903	2903	2903	2903
Arithmetic mean	30 287 022	811892.7975	4650630.875	13640073.42	4.44%	13.59%	9.83%
Standard deviation	61 865 463	5875009.556	15684490.05	16764626.11	-	-	-
Median	14 022 067	212 816	267 921	7 297 216	2.81%	13.24%	9.40%
winsorized mean	**17 592 490**	**316 503**	**1 385 844**	**9 218 665**	2.98%	13.58%	9.38%
Truncated mean	54 655 970	1 225 279	5 823 678	28 821 327	3.54%	13.75%	9.53%
Skewness	**9**	**45**	**8**	**2**	-	-	-
Maximum	1 512 338 750	299 344 737	229 794 569	98 868 194	-	-	-
Minimum	- **166 617 306**	- **27 499 000**	**-**	**374**	-	-	-

The empirical data from Polish firms for 2009-2010 years suggests that for Polish managing teams risk aversion has stronger influence on liquid assets investment policy than cost of capital.

Figure 12. The expected change in liquid assets indicators after changes in risk indicators. Source: own study.

2009	Indicator	Change	2010
0.82	β	↘	0.83
10,00%	CC	↘	9,40%
0.36	CA/Rev	↗	0.37
0.51	CA/Assets	-	0.51
8.82	CA/EBIT	↗	9.28

The illustration of that influence is presented in Figure 12.

SUMMARY AND CONCLUSIONS

Depending on the business type that the given enterprise is doing, sensibility to liquid assets financing method risk might vary a lot. Character of business also determines the best strategy that should be chosen whether it will be the conservative strategy (situation closer to the first variant) or aggressive one (situation closer to the first variant) or maybe some of the transitional variants similar to the Compromise strategy. The best choice is that with the adequate cost of financing and highest enterprise value growth. This depends on the structure of financing costs. The lower the financing cost, the higher effectiveness of enterprises activity measured by the growth of its value. The firm choosing between various solutions in liquid assets needs to decide what level of risk is acceptable for her owners and capital suppliers. It was shown in solutions presented in that paper. If the risk aversion is higher, will be preferred more safe solution. That choice results with cost of financing consequences. In this paper, we considered that relation between risk and expected benefits from the liquid assets decision and its results on financing costs for the firm. The empirical data from Polish firms for 2009-2010 years suggests that for Polish managing teams risk aversion has stronger influence on liquid assets investment policy than cost of capital.

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INVESTIGATING DIFFERENCES BETWEEN THE CZECH AND SLOVAK LABOUR MARKET

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Abstract

This contribution reveals some structural properties and differences of the Czech and Slovak labour markets. A small search and matching model incorporated into standard DSGE model is estimated using Bayesian techniques. Two sources of rigidities were implemented: wage bargaining mechanism and "search and matching" process matching workers and firms. The results show that search and matching aspect provides satisfactory description of employment flows in both economies. Model estimates provide interesting evidence that wage bargaining process is determined mainly by the power of the firms. These results support the view of flexible wage environment in both economies. On the other hand, the firms are confronted by the increasing vacancy posting costs that limit vacancies creation. Relative low separation rate provides evidence of reduced mobility of the workers.

Keywords: *search and matching model, DSGE model, wage bargaining, Bayesian estimation*

JEL Classification: C51, E24, J60
AMS Classification: 91B40, 91B51

1 INTRODUCTION

Labour market and its structural properties are the key determinants of the business cycles fluctuations. The goal of my contribution is to reveal some interesting and important structural differences of the Czech and Slovak labour markets in the last twelve. For this purpose, I use a small search and matching model incorporated into standard macroeconomic dynamic stochastic general equilibrium model (DSGE). Search and matching model is an important tool to model labour market dynamics. This model is a log-linear version of the model originally developed by Lubik [8]. Using real macroeconomic data I am able to estimate some key labour market indicators: the wage bargaining power of unions, the match elasticity of unemployed and the efficiency of the matching process.

One of the main questions of this paper is how flexible are the Czech and Slovak Labour markets. There is not a unique measure of the labour market flexibility but one can focus on some key features which might be connected with a flexible labour market. In case of the Slovak labour market, Gertler [5] studies the relationship between the local unemployment rate and wage level (using a panel data approach). He has confirmed that wages in Slovakia are relatively flexible (that is an important part of labour market flexibility concept). But, this overall wage flexibility was only poorly influenced by the institutional arrangements of the Slovak labour market. The labour market in the Czech Republic was influenced (like the Slovak labour market) by the opening of markets which started in 1990. As Flek and Večerník [3] pointed out, the market reforms, trade and price liberalisation and the establishment of standard labour market institutions (aiming on improvement of labour mobility and flexibility) produced an inevitability of rising unemployment. Unlike other transition countries (including Slovakia) the rise of unemployment was delayed and unemployment rate hitted 10-years peak in 2004. Flek and Večerník [3] argue that the labour market alone was not fully responsible for this poor performance. Some obstacles (to better macroeconomic performance and job creation) were linked with a relatively weak supply-side flexibility of the Czech economy as a whole. These authors conclude that the Czech labour market loses its flexibility due to high reservation wage and due to the obstacles connected with the necessary layoffs. This conclusion is confirmed by

Gottvald [6]. On the other hand, he pointed out that the diminishing flexibility in 90s was accompanied by the high probability of changing job (without an episode of unemployment). He observed decreasing flows of workers among industries (i.e. low labour mobility). Other aspect of the Czech labour market are analysed by Mareš and Sirovátka [9] which emphasized the role of long-term unemployment (a problem faced by the Slovak labour market as well). Wage flexibility (on regional level) was discussed by Galuščák and Münich [4]. I am convinced that some of these issues may be confronted with the results of presented DSGE model.

2 MODEL AND DATA

As mentioned previously, I use the model developed by Lubik [8]. It is a simple search and matching model incorporated within a standard DSGE framework. The labour market is subject to friction because a time-consuming search process for workers and firms. The wages are determined by the outcome of a bargaining process which serves as a mechanism to redistribute the costs of finding a partner. For estimation purposes, I did not use the non-linear form of the model mentioned in the previous section (of course, this form is important to understand the meaning of the key structural model parameters). Instead of that, I use a log-linear version of the model based on my own derivations. Log-linear version is not a part of the original contribution of Lubik [8] and may be found in Němec [10].

The model for the Czech and the Slovak economy is estimated using the quarterly data set covering a sample from 1999Q1 to 2010Q4. The observed variables are real output (GDP, in logs), hourly earnings (in logs), unemployment rate and rate of unfilled job vacancies. All data are seasonally adjusted. The original data are from databases of the OECD, the Czech Statistical Office (CZSO) and the Ministry of Labour, Social Affairs and Family of the Slovak Republic (SAFSR) and the Statistical Office of the Slovak Republic (SOSR).[1] Real output and hourly earnings are de-trended using Hodrick-Prescott filter (with the smoothing parameter $\lambda = 1600$ $). The rate of unfilled job vacancies and unemployment rate was demeaned prior estimation. The variables used are expressed as corresponding gaps. It should be mentioned, that the unemployment gap and the gap of vacancies were computed as log differences (i.e. both series and their means were expressed in logarithms before differencing). This approach is consistent with the log-linear equations (see Němec [10]). The estimation results are thus different (in some ways) from the ones presented by Němec [10] who used simply the corresponding differences.

3 ESTIMATION RESULTS AND MODEL EVALUATION

Parameters are estimated using Bayesian techniques combined with Kalman filtering procedures. All computations have been performed using Dynare toolbox for Matlab (Adjemian et al. [1]).

Table 1 reports the model parameters and the corresponding prior densities. The priors (and calibrations) are similar to those used by Lubik [8]. On the other hand, the standard deviations are rather uninformative.

Table 1: Parameters and prior densities

Description	Parameter	Density	Priors SVK		Priors CZE	
			Mean	Std. Dev.	Mean	Std. Dev
Discount factor	β	-	0.99	-	0.99	-
Labour elasticity	α	-	0.67	-	0.67	-
Demand elasticity	ε	-	10	-	10	-
Relative risk aversion	σ	G	1.00	0.50	1.00	0.50
Match elasticity	ξ	G	0.70	0.10	0.70	0.10
Separation rate	ρ	G	0.10	0.05	0.10	0.05

[1] I used the following data sets: GDP at purchaser prices, constant prices 2000, s.a., CZSO, millions of CZK; GDP at purchaser prices, constant prices 2000, s.a., SOSR, millions of EUR; Index of hourly earnings (manufacturing), 2005=100, s.a., OECD; Registered unemployment rate, s.a., OECD; Unfilled job vacancies, level (transformed to ratio of unfilled vacancies to labour force), s.a., OECD and SAFSR.

Bargaining power of the workers	η	U	0.50	0.30	0.50	0.30
Unemployment benefits	b	B	0.20	0.15	0.20	0.15
Elasticity of vacancy creation cost	ψ	G	1.00	0.50	1.00	0.50
Scaling factor on vacancy creation cost	κ	G	0.10	0.05	0.10	0.05
AR coefficients of shocks	$\rho_{\chi,A,\mu,Y}$	B	0.80	0.20	0.80	0.20
Standard deviation of shocks	$\sigma_{\chi,A,\mu}$	IG	0.01	1	0.01	1.00
Standard deviation of shocks	σ_Y	IG	0.05	1	0.05	1.00

Chyba! Nenašiel sa žiaden zdroj odkazov. presents the posterior estimates of parameters and 90\% highest posterior density intervals. It may be seen (in comparison with the
Table 1) that most of the parameters are moved considerably from their prior means. The data seems to be strongly informative.
There are some remarkable results which should be emphasized. The first surprising estimate is the bargaining power of workers, η. The mean value of this parameter is almost 0 for both countries with a 90 percent coverage region that is shifted considerably away from the prior density. This implies that the firms can gain the most of their entire surplus. The firms are thus willing to create vacancies. This result is in accordance with to the results of Lubik [8] or Yashiv [11] who aimed to model the U.S. labour market. Low bargaining power of the workers is typical for the flexible labour markets which bring the wage dynamics to the line with productivity growth. The second interesting result is the estimated separation rate, ρ. This parameter is considerably lower than the one estimated by Lubik [8]. Its value supports the view of less flexible Czech and Slovak labour market with limited ability to destroy old and new matches. Low flexibility is meant to be associated with the restricted flows of the workers among industries.

Table 2: Parameters estimates

	SVK			CZE		
	Posterior mean	90% HPDI		Posterior mean	90% HPDI	
σ	0.2843	0.1319	0.4248	0.4517	0.2989	0.5648
ξ	0.8196	0.7645	0.8782	0.7758	0.7229	0.8316
ρ	0.0677	0.0185	0.1259	0.0705	0.0563	0.0843
η	0.0046	0.0000	0.0099	0.0022	0.0000	0.0050
b	0.1566	0.0001	0.2988	0.4557	0.4083	0.5052
ψ	2.2769	1.7870	2.7440	1.9257	1.8313	2.0563
κ	0.1245	0.0811	0.1759	0.0875	0.0524	0.1259
ρ_χ	0.2514	0.0616	0.4554	0.7347	0.6994	0.7641
ρ_A	0.9449	0.8785	1.0000	0.9851	0.9802	0.9914
ρ_μ	0.9563	0.9188	0.998	0.8222	0.7211	0.8804
ρ_Y	0.8079	0.6948	0.9267	0.9184	0.8632	0.9806
σ_χ	0.0170	0.0141	0.0199	0.0085	0.0071	0.0099
σ_A	0.5063	0.1300	0.8161	0.3181	0.2429	0.3981
σ_μ	0.0640	0.0531	0.0743	0.0666	0.0551	0.0767
σ_Y	0.0168	0.0142	0.0194	0.0097	0.0082	0.0112

The third remarkable estimate is the vacancy posting elasticity, ψ. The posterior means 2.3 for the Slovak labour market and 1.9 for the Czech labour market are shifted away from the prior mean. The vacancy creation is thus more costly because of increasing marginal posting costs (increasing in the level of vacancies or labour market tightness, $\theta = v_t / u_t$). Lubik [8] estimated this parameter at the mean value of 2.53. In this case, the high value of ψ may be interpreted as

a balancing factor which "restrict"' potentially excessive vacancy creation driven by the low bargaining power. In case of the analysed labour markets, this higher value provides further evidence of specifically less flexible labour markets. The estimate of parameter b corresponds to a remarkably high value of 0.46 for the Czech economy which might be in accordance with the real unemployment benefits paid within the Czech social insurance system (40% of average wage). The lower value of 0.16 for the Slovak economy might support the view of lower reservation wage for this country. The posterior mean of the matching function parameter, ξ, is in accordance with the common values in literature (see Lubik [8] or Christoffel et al. [2]).
The trajectories[2] of selected smoothed variables and shock innovations show a relative sharp decline in the development of variable q (probability of filling a vacancy) at the end of the year 2006. This evidence is in favour of theories which stressed the role of an obvious lack of employees in the Czech economy. The similar results may be found for the Slovak economy as well. This tendency was reverted as a result of the last global economic slowdown starting at the end of 2008. This downturn of both economies influenced a fall of the matching rates (m) below their steady-state values. On the other hand, the starting recession has re-established the equilibrium on both labour markets (see the trajectories of employment rate and labour market tightness). The improvement of labour market institutions might be associated with the development of efficiency shock (μ). From this point view, some remarkable changes on the Czech and Slovak labour markets started at the end of 2004 and at the beginning of the 2006 respectively. Historical shock decomposition reveals the fact that the technology shock plays more important role in the Czech economy. A story about the properties and development of both labour markets is similar to the one discussed previously.
In order to see how the model fits the data, sample moments, autocorrelation coefficients and cross-correlations were computed. I computed these statistics from simulation of the estimated models with parameters set at their posterior means. All these statistics correspond to the four observed series (unemployment gap, u, gap of vacancies, v, gap of the wages, w, and output gap, Y). Both models are successful in matching all sample moments and autocorrelation coefficients (they are mostly within the appropriate 90% highest posterior density intervals). This ability is not used to be typical for such a small-scale model. Unlike the results of Němec [10], there is no exception regarding the fit of sample moments. The model using the data for unemployment gap and vacancies gap as log-differenced variables does not predict volatility in wages higher as observed. My results are in accordance with the authors arguing that the model with search and matching frictions in the labour market is able to generate negative correlation between vacancies and unemployment (see Krause and Lubik [7]). Unfortunately, the values of cross-correlation coefficients are not sufficient for the correlations of wages and the rest of observable variables (especially in the case of the Czech model). The similar experience may be found in the results for U.S. labour market provided by Lubik [8]. Lubik pointed out that this may be due the presence of matching shock, which can act as a residual in employment and wage equations.

4 CONCLUSION

In my contribution, I investigated structural properties of the Czech and Slovak labour markets using a simple DSGE framework with labour market frictions. Two sources of rigidities were implemented: wage bargaining mechanism and "`search and matching"' process matching workers and firms. Estimated model provides satisfactory description of employment flows in both economies. Parameter estimates provide convincing evidence that wage bargaining process is determined mainly by the power of the firms. The structural properties of both markets do not differ too much from the properties of the U.S. labour market.

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[2] Due to the maximal allowed range of the contribution, all the figures (data, smoothed variables and shock, IRFs and shock decompositions) are a part of accompanying conference presentation and may be obtained upon a request.

As for the labour markets flexibility, my results support the view of flexible wage environment in both economies. On the other hand, the firms are confronted by the increasing vacancy posting costs that limit vacancies creation. Moreover, the lower separation rate might provide us with the evidence of reduced mobility of the workers. Unfortunately, because of simple structure of the model presented in this paper, there are some drawbacks which should be mentioned and which are connected with some suggestions for further research: robustness check based on estimation using the information provided by a variety of filters or by direct linking of the observable data to the DSGE models, inclusion of price rigidities and monetary policy (monetary rule) which allows to analyse implications of wages and labour market shocks on inflation process and incorporating labour market rigidities into an open economy model because foreign demand should play a significant role in the development of both economies (the direct effects of labour market shocks will become more obvious).

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A SIMPLE SOLUTION OF A SPECIAL QUADRATIC PROGRAMMING PROBLEM

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Abstract
This paper studies a special quadratic programming problem arising from a practical problem – minimization of passenger waiting time on a bus stop. Several properties of an optimum solution of this problem are proved. A graph theory algorithm based on proved properties is proposed. The designed algorithm does not make use of traditional methods of quadratic programming.

Keywords: *quadratic programming, directed graph, shortest path*

JEL Classification: C02
AMS Classification: 65K05, 90C20, 05C38, 05C85

1 INTRODUCTION

Černý in [1] introduced the problem of optimum time distribution of trips with respect to the total passenger waiting time. Passengers are arriving to a bus stop of a line within the time interval $\langle t_1 t_2 \rangle$ with density $f(t)$. Total number of passengers arriving to the bus stop during whole interval $\langle t_k t_{k+1} \rangle$ can be expressed as $\int_{t_k}^{t_{k+1}} f(t)dt$ and total waiting time can be calculated as $\int_{t_k}^{t_{k+1}} f(t)(t-t_1)dt$. Suppose that $t_0, t_1, \dots, t_n$ are departures of all trips from the considered bus stop sorted in ascending order. The total waiting time of all passengers during whole day is

$$W(t_0, t_1, \dots, t_n) = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(t)(t-t_k)dt \qquad (1)$$

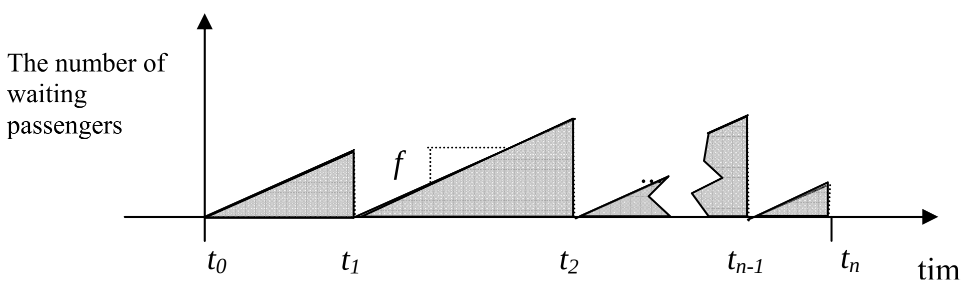
Suppose that the departure times t_0, t_n of the first and the last trip are fixed. The problem is to minimize the function $W(t_0, t_1, \dots, t_n)$, i. e. to find $t_0, t_1, \dots, t_n$ such that $t_0 < t_1 < \dots < t_n$ such that the objective $W(t_0, t_1, \dots, t_n)$ is minimal. Černý proved that this problem fulfills Bellman's principle of optimality and designed a procedure for solving this problem using dynamic programming techniques.

Janáček, Koháni, Kozel and Gábrišová in [2], [3] and [4] noticed, that practice requires that departure times can be shifted only in limited intervals and therefore they have solved similar problem with additional constraints expressed as $t_i \in \langle a_i, b_i \rangle$ for $i = 1,2,\dots,n-1$ for constant passenger density function, i.e. $f(t) = const$. In this case

$$W(t_0, t_1, \dots, t_n) = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f.(t-t_k)dt = \sum_{k=0}^{n-1} \frac{1}{2} f.(t_{k+1}-t_k)^2 \qquad (2)$$

The following figure (taken over from [3]) illustrates the total passenger waiting time by shaded area.

Figure 1 The waiting time of passengers during period <t_0, t_n>.



Just described problem leads to the following mathematical model:

$$\mathit{Minimize} \qquad \sum_{k=0}^{n-1}(x_{k+1} - x_k)^2 \tag{3}$$

$$\mathit{subject\ to} \qquad a_i \le x_i \le b_i \quad \mathit{for}\ i = 0,1,2,\dots,\mathrm{n} \tag{4}$$

The conditions that $x_0 = t_0$ and $x_n = t_n$ can be satisfied by setting a_0=b_0=t_0 and a_n=b_n=t_n . This is a quadratic programming problem (QPP). Gábrišová and Kozel tried to solve this problem by linear approximation used in non-linear problem optimization. This paper shows that it can be solved by very simple mathematical tools.

2 PROPERTIES OF AN OPTIMUM SOLUTION

Let u, v are two real numbers, $u < v$. Let us have a function $h(x)$ defined on interval $\langle u, v\rangle$ by formula

$$h(x) = (x-u)^2 - (v-x)^2 \tag{5}$$

Then the function $h(x)$ has a global minimum at $x = \frac{u+v}{2}$. Moreover, the function $h(x)$ is strictly decreasing on interval $\langle u, \frac{u+v}{2}\rangle$ and strictly increasing on interval $\langle \frac{u+v}{2}, v\rangle$.

Clearly, the first derivative of $h(x)$ is $h'(x) = 4x - 2(u+v)$, the second derivative

of $h(x)$ is $h''(x) = 4$. It holds: $h'(\frac{u+v}{2}) = 0$, $h'(x) < 0$ on interval $\langle u, \frac{u+v}{2}\rangle$ and $h'(x) > 0$ on interval $\langle \frac{u+v}{2}, v\rangle$.

Theorem 1. Let $(x_0^*, x_1^*, \dots, x_n^*)$ be an optimum solution of quadratic programming problem (3), (4). Then $x_0^* = t_0$, $x_n^* = t_n$ and it holds for x_i^* where $0 < i < n$:
Either $x_i^* = a_i$ or $x_i^* = b_i$ $x_i^* - x_{i-1}^* = x_{i+1}^* - x_i^*$, i.e.

$$x_i^* = \frac{x_{i+1}^* - x_{i-1}^*}{2} \tag{6}$$

This means that if x_i^* is different from endpoints of interval $\langle a_i, b_i\rangle$ then x_i^* is the center of interval $\langle x_{i-1}^*, x_{i+1}^*\rangle$.

Proof by contradiction.
Suppose $x_i^* \in (a_i, b_i)$ (i. e. $x_i^* \ne a_i$ and $x_i^* \ne b_i$). Suppose $x_i^* - x_{i-1}^* \ne x_{i+1}^* - x_i^*$.
In this case x_i^* can be shifted towards the center of interval $\langle x_{i-1}^*, x_{i+1}^*\rangle$ by a little value ε. The consequence of above proved properties of function $h(x)$ implies that the outcome of such a shift results in an increase of the sum $(x_i^* - x_{i-1}^*)^2 + (x_{i+1}^* - x_i^*)^2$ what implies an increase of objective function $\sum_{k=0}^{n-1}(x_{k+1}^* - x_k^*)^2$.

Therefore a solution of quadratic programming problem (3), (4) with some $x_i^* \in (a_i, b_i)$ such that $x_i^* - x_{i-1}^* \ne x_{i+1}^* - x_i^*$ (i.e. such that $x_i^* \ne \frac{x_{i+1}^* - x_{i-1}^*}{2}$) cannot be an optimum solution.
Let $(x_0^*, x_1^*, \dots, x_n^*)$ be a feasible solution of quadratic programming problem (3), (4).
Variable x_i^* is called **bounded variable** if $x_i^* = a_i$ or $x_i^* = b_i$.
Variable x_i^* is called **inner variable** if $x_i^* \in (a_i, b_i)$.

Theorem 2. Every optimum solution of quadratic programming problem (3), (4) is in the form

$$\boldsymbol{x} = x_{k_0} = t_0 = x_0, x_1, \dots, x_{k_1-1}, x_{k_1}, x_{k_1+1}, \dots, x_{k_2-1}, x_{k_2}, x_{k_2+1}, \dots, x_{k_2-1}, x_{k_2}, x_{k_2+1}, \dots$$

$$\dots, x_{k_3-1}, x_{k_3}, x_{k_3+1}, \dots, \dots, x_{k_{r-1}-1}, x_{k_{r-1}}, x_{k_{r-1}+1}, \dots, x_{k_r} = x_n = t_n \tag{7}$$

where $0 = k_0 < k_1 < \cdots < k_r = n$, where x_{k_i} is a bounded variable for $i = 0, 1, \dots, r$, where all other variables are inner variables and if $k_{i+1} > k_i + 1$ then

$$x_{k_i+j} = x_{k_i} + j.\frac{x_{k_{i+1}} - x_{k_i}}{k_{i+1} - k_i} \quad \text{for } j = 1, \dots, (k_{i+1} - k_i - 1) \tag{8}$$

Theorem 2 says that every optimum solution of QPP (3), (4) is determined by a sequence $x_{k_0}, x_{k_1}, \dots, x_{k_r}$, where $x_{k_i} = a_{k_i}$ or $x_{k_i} = b_{k_i}$.
Moreover, the sequence $x_{k_0}, x_{k_1}, \dots, x_{k_r}$ has to fulfill the following condition:

Condition 1. For all points of the type

$$x_{k_i+j} = x_{k_i} + j.\frac{x_{k_{i+1}} - x_{k_i}}{k_{i+1} - k_i} \quad \text{for } j = 1, \dots, (k_{i+1} - k_i - 1) \text{ it holds}$$

$$x_{k_i+j} \in (a_{k_i+j}, b_{k_i+j}). \tag{9}$$

Therefore an optimum solution of QPP (3), (4) can be found by exploring finite number of feasible subsequences of sequence $a_0, b_0, a_1, b_1, \dots, a_n, b_n$, starting with a_0, finishing with b_n and containing at most one element from every pair $a_i,\ b_i$.

Let $G = (V, A, c)$ be a directed graph with vertex set $V = \{a_0, b_0, a_1, b_1, \dots, a_n, b_n\}$ and arc set $A = A_1 \cup A_2 \cup A_3 \cup A_4$ where

$$A_1 = \Big\{(a_i, a_j) \mid i < j, \quad i, j = 0,1, \dots n, \quad a_i + k\frac{a_j - a_i}{j - i} \in \langle a_{i+k}, b_{i+k} \rangle \ \forall k = 1,2, \dots j - i - 1\Big\}$$

$$A_2 = \Big\{(a_i, b_j) \mid i < j, \quad i, j = 0,1, \dots n, \quad a_i + k\frac{b_j - a_i}{j - i} \in \langle a_{i+k}, b_{i+k} \rangle \ \forall k = 1,2, \dots j - i - 1\Big\}$$

$$A_3 = \Big\{(b_i, a_j) \mid i < j, \quad i, j = 0,1, \dots n, \quad b_i + k\frac{a_j - b_i}{j - i} \in \langle a_{i+k}, b_{i+k} \rangle \ \forall k = 1,2, \dots j - i - 1\Big\}$$

$$A_4 = \Big\{(b_i, b_j) \mid i < j, \quad i, j = 0,1, \dots n, \quad b_i + k\frac{b_j - b_i}{j - i} \in \langle a_{i+k}, b_{i+k} \rangle \ \forall k = 1,2, \dots j - i - 1\Big\}$$

The cost of arc $(u, v) \in A$ where $u = a_i$ or $= b_i$, $v = a_j$ or $v = b_j$ and $i < j$ is defined as

$$c(u, v) = (j - i)\left(\frac{v-u}{j-i}\right)^2 = \frac{(v-u)^2}{j-i}. \tag{10}$$

The cost $c(u, v)$ (10) represents the part $\sum_{k=i}^{j-1}(x_{k+1} - x_k)^2$ of objective function (3) for $x_i = u$, $x_j = v$ and $x_k = u + k\frac{v-u}{j-i}$ for $k = 1,2, \dots j - i - 1$.

Therefore if $\boldsymbol{x}$ is an optimum solution of (3), (4) in the form (7), then all ordered pairs $(x_{x_0}, x_{k_1}), (x_{x_1}, x_{k_2}), \dots, (x_{x_{r-1}}, x_{k_r})$ are the arcs of directed graph $G = (V, A, c)$ and the sequence

$$\boldsymbol{\mu}(a_0, b_n) = a_0 = x_{x_0}, (x_{x_0}, x_{k_1}), x_{k_1}, (x_{x_1}, x_{k_2}), x_{k_2}, \dots, x_{x_{r-1}}, (x_{x_{r-1}}, x_{k_r}), x_{k_r} = b_n \tag{11}$$

is a path in $G = (V, A, c)$ having the length equal to the objective value of $\boldsymbol{x}$.

On the other hand every path $\boldsymbol{\mu}(a_0, b_n)$ in $G = (V, A, c)$ defines a feasible solution $\boldsymbol{x}$ of QPP (3), (4) with objective function value equal to length of the path $\boldsymbol{\mu}(a_0, b_n)$.

Corollary 1. To find an optimum solution of QPP (3), (4) means to find a shortest path in directed graph $G = (V, A, c)$.

Let us have the following enumeration of vertices of $G = (V, A, c)$:

a_0	b_0	a_1	b_1	$\dots$	$\dots$	a_{n-1}	b_{n-1}	a_n	b_n
v_0	v_1	v_2	v_3	$\dots$	$\dots$	v_{2n-2}	v_{2n-1}	v_{2n}	v_{2n+1}

Algorithm. Shortest $(v_0 - v_{2n+1})$-path algorithm customized for digraph $G = (V, A, c)$:
Step 1. Assign two labels $t(v_i)$, $x(v_i)$ to every $v_i \in V$
Set $t(v_0) = 0$, $y(v_0) = 0$.
Set $y(v_i) = 0$, $t(v_i) = \infty$ for all $i = 1, 2, \dots, 2n + 1$.
Step 2. For $i = 0, 1, \dots, 2n$ do:
For all w such that $(v_i, w) \in A$ do:
If $t(w) > t(vi) + c(v_i, w)$
then set $t(w) = t(v_i) + c(v_i, w)$, and $y(w) = v_i$.

When algorithm finishes the optimum value of objective function is equal to $t(v_{2n+1})$ and the corresponding optimum solution of QPP (3), (4) is defined by it's bounded points

$$v_{2n+1},\ y(v_{2n+1}), y\left(y(v_{2n+1})\right), y\left(y\left(y(v_{2n+1})\right)\right), \dots, y\left(\dots y\left(y(v_{2n+1})\right)\dots\right) = v_0,$$

3 COMPUTATIONAL RESULT

a_i	b_i	x_i^* Exact solution obtained by proposed algorithm	solution obtained by built in solver in MS Excel 2007
0	0	**0**	0
10	25	**25**	25
70	90	76,25	76,25715
120	140	127,5	127,5101
170	200	178,75	178,7571
100	230	**230**	230
300	330	322,5	322,4127
350	420	415	414,9111
500	520	507,5	507,4126
600	610	**600**	600
620	630	625	624,9997
630	650	**650**	650
700	730	725	724,9996
800	810	**800**	800
830	850	850	850
860	900	**900**	900
910	1100	1016,666687	1016,667
1120	1180	1133,333374	1133,336
1190	1250	**1250**	1250
1400	1400	**1400**	1400

The value of objective function of our exact solution is 126189.586, the corresponding result given by MS Excel solver is 126189.599.

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PORTFOLIO RETURN, TAKING INTO ACCOUNT THE COSTS OF FINANCIAL TRANSACTIONS

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Abstract
To address the role of portfolio selection, various mathematical programming models could be used. The paper presents a model based on Conditional value at risk (CVaR) due to its good qualities and its suitability for testing the presented problem. The analyzed problem is aimed to compare the net portfolio return (taking into account the cost of portfolio change) in the case that during the reporting time the investor's portfolio that is based on information obtained prior to the period examined, remain unchanged, or possibly the portfolio is generated also on the base of information that are obtained in the course of the analyzed period.

Keywords: *CVaR, portfolio selection, return*

JEL Classification: C44
AMS Classification: 90C15

Introduction
In this paper the authors present the issue of portfolio choice in a given timeframe, taking into account the cost of the portfolio change. The first section describes the chosen risk level and the corresponding model of portfolio selection that is based on that rate. The authors chose the very popular Conditional value the risk (CVaR). The advantage of this rate is, except its other qualities, the possibility of formulation the problem of the portfolio selection that does not require the conversion of input parameters, but only the embedment of the condition based on corresponding time series is involved that is given in the second section of the article. The last section contains the analyze that is based on DivDax stock index, where the objective was to compare the corresponding portfolio return in the case that the investor's portfolio remains calculated on the base of historical data and in a situation that investor's portfolio is changed after receiving new information about the stock return.

1 CONDITIONAL VALUE AT RISK (CVAR)

Value at risk (VaR) has become a standard tool of risk management in the financial sector, mainly because of its conceptual and computational simplicity. However, according to many authors, the use of VaR brings several conceptual problems. The main critics of VaR are e.g. Artzner (1999) and Basak - Shapiro (2001). For example, Artzner (1997, 1999) noted the following shortcomings of VaR:

1. VaR measures only percentiles of profit and loss, and thus disregards the loss beyond the VaR,
2. VaR is not a coherent risk measure because it is not sub-additive.

An alternative measure of risk called conditional value at risk (CVaR) was proposed by Artyner (1997) to eliminate the problems posed by VaR. CVaR is a coherent measure of risk, and therefore the problems of portfolio selection can be formulated as linear programming problem.
CVaR is defined as the dangers beyond VaR. Based on this idea, CVaR value takes into account loss over the value of VaR. The advantage of CVaR is its sub-additivity, which ensures consistency (coherence) as a measure of risk. Based on these facts, some investors tend to the use of CVaR instead VaR.
Information provided by VaR may mislead investors who want to maximize its benefit. Therefore, the use of VaR, as the only measure of risk is not an appropriate approach and

therefore it could lead to the conclusion that would bring the loss. Investors can avoid the above problems by adopting CVaR as a conceptual point of view that takes into account dangers beyond VaR. The effectiveness of the expected value at risk depends on the stability of the estimates and also on the choice of efficient backtesting methods.

2 CVAR AS LINER PROGRAMMING MODEL

Based on the previous section it can be constructed the model of portfolio selection. In this section the construction of the model on the base of CVaR as the linear programming model will be discussed.

The goal is to find a value of optimal weights, whose allow minimizing value at risk that is measured by CVaR. The corresponding optimization problem (Zenios, 2007):

$$CVaR_\alpha(X) = min\left\{VaR_\alpha + \frac{1}{\alpha}E\left[\left(E_p - X - VaR_\alpha\right)^+\right]\right\}, \quad (1)$$

where VaR_α - Value at risk, E_p - target return, formula $\left(E_p - X - VaR\right)^+$ is a positive part of difference $E_p - X - VaR$.

Suppose portfolio that consist of n stocks, of the vector of weights $\boldsymbol{w} = \left(w_1, w_2, \dots, w_n\right)$ and of the vector of return for k-th variant of individual stock $\boldsymbol{r_k} = \left(r_{1,k}, r_{2,k}, \dots, r_{n,k}\right)$. Thus the expectant return of portfolio could be calculated as $\boldsymbol{w}^T\boldsymbol{r_k}$ and expected return is mean value $E[\boldsymbol{w}^T\boldsymbol{r_k}]$. Than the relation (1) could be formulated as

$$CVaR_\alpha(\boldsymbol{w}^T\boldsymbol{r}) = min\left\{VaR_\alpha + \frac{1}{\alpha}E\left[\left(E_p - \boldsymbol{w}^T\boldsymbol{r_k} - VaR_\alpha\right)^+\right]\right\}. \quad (2)$$

Suppose that the possible variants arise with the same probability. Further on let t be the number of rows of matrix of variant of individual stocks. Then the objective function could be formulated as:

$$min\left\{VaR_\alpha + \frac{1}{\alpha t}\sum_{k=1}^{t}\left[E_p - \boldsymbol{w^T}\boldsymbol{r_k} - \mathrm{VaR}_\alpha\right]^+\right\} \quad (3)$$

To avoid the nonlinear formulation it is necessary to replace the element $\left[E_p - \boldsymbol{w}^T\boldsymbol{r_k} - VaR_\alpha\right]^+$ with the variable $\boldsymbol{z} = (z_1, z_2, \dots, z_t)$, where $z_k \geq 0$. The linear programming formulation using the variable $z_k \geq \left[E_p - \boldsymbol{w}^T\boldsymbol{r_k} - VaR_\alpha\right]^+$ could be written as follows:

$$min\left\{VaR_\alpha + \frac{1}{\alpha t}\sum_{k=1}^{t} z_k\right\} \quad (4)$$

$$z_k - E_p + \boldsymbol{w^T}\boldsymbol{r_k} + VaR_\alpha \geq 0, \quad k = \{1,2,\dots,t\}, \quad (5)$$

$$\boldsymbol{w^T}\boldsymbol{E}(\boldsymbol{r_n}) \geq E_p, \quad (6)$$

$$w^T e = 1, \quad (7)$$

$$z \geq 0, \quad (8)$$

where $\boldsymbol{E}(\boldsymbol{r_n})$ is vector of expected returns of assets.

3 ANALYSIS OF PORTFOLIO CHOICE TAKING INTO ACCOUNT THE COST OF PORTFOLIO CHANGE

As was mentioned before, the article is focused on the analysis of returns taking into account the change in portfolio in the short period. The model was constructed in two-dimensional space (the expected returns and CVaR) and we tested the portfolio selection taking into account the cost of changing the portfolio to compare it to a situation that an investor holds a portfolio constructed independently of the latest information about assets.

The analyzes was provided according to subjects, whose shares are traded on German stock market (Deutsche Börse). The paper used daily data of the following 15 companies[1]: ALLIANZ SE VNA O.N., BASF SE NA O.N., BAY.MOTOREN WERKE AG ST, BAYER AG NA, DAIMLER AG NA O.N., DEUTSCHE BOERSE NA O.N., DEUTSCHE POST AG NA O.N.,

[1] http://www.dax-indices.com/DE/index.aspx?pageID=4

DT.TELEKOM AG NA, E.ON AG NA, LUFTHANSA AG VNA O.N., MAN SE ST O.N., METRO AG ST O.N., MUENCH.RUECKVERS.VNA O.N., RWE AG ST O.N., SIEMENS AG NA, which assets are parts of the stock index DivDax. In this paper we use that stock index to determine the importance of the portfolio change during a short period (19.09.2011-03.09.2012). After testing we observed that it is not clear if it is worth to modify the composition of investment portfolios over time, or hold a portfolio throughout the entire investment.

In the given example, the investment period was set to 10 trading days on the market (from 27.02.2012 to 09.03.2012), and we use available daily historical data on the value of shares in the stock market. For the purposes of our calculations, we transform them into a daily income and on that base we set the estimated revenue for each day of investments that we also serve to determine composition of the portfolio using the model in space CVaR and expected return (Table 1).

Table 1 Portfolio composition on the base of CVaR model; Source: The authors

	BASF SE NA O.N.	BAYER AG NA	DEUTSCHE BOERSE Z.UMT.	DEUTSCHE POST AG NA O.N.	DT.TELEKOM AG NA	MAN SE ST O.N.	MUENCH. RUECKVERS.VNA O.N.	SIEMENS AG NA
27.2.2012	0	0.111771	0.3128075	0	0.29299712	0	0.282424207	0
28.2.2012	0	0.10152	0.3191005	0	0.30246744	0	0.276911906	0
29.2.2012	0	0.10029	0.3198556	0	0.3036038	0	0.276250478	0
1.3.2012	0.0044	0	0.1028787	0.1541566	0.26657312	0	0.471994874	0
2.3.2012	0	0	0.1011441	0.15614411	0.26890065	0	0.472524211	0.001287
5.3.2012	0	0	0	0.29924455	0.21359285	0	0.38932176	0.097841
6.3.2012	0	0	0	0.29918924	0.22193743	0	0.415830132	0.063043
7.3.2012	0	0.022973	0	0.44677518	0.16507041	0.011655	0.353526502	0
8.3.2012	0	0.046284	0	0.4027964	0.1637237	0	0.387195942	0
9.3.2012	0	0	0	0.41244063	0.11387588	0	0.275135302	0.198548

As can be seen in Table 1, based on the model in the space of the expected return and of the CVaR, the portfolio is changed over time. These changes are related to the equity returns of individual firms and to the risks that are associated with these revenues. Most of the portfolio share was successively deposited into shares of DEUTSCHE POST AG NA O.N. that prove relatively high return at relatively low risk.

The resulting portfolio (Table 1) was calculated on the base of input parameters: the target daily return 0.2% and 5% significance level. Portfolio weights of individual components were affected by long-term nature of the data. To a final portfolio were included primarily the shares of companies that achieve the highest return at relatively low risk, the shares with a higher return but also at high risk have been excluded from the portfolio. Changes in the composition of portfolio were caused by price changes of shares of companies that increased the risk of an investment portfolio and therefore they have no longer fulfill the conditions of required return on given level of risk. Overall results and comparison of benefit of portfolio change during the period of the investment are summarized in Table 2.

As is shown in Table 2, the return of dynamically evolving portfolio and a portfolio whose composition is unchanged throughout the period are identical in the first period because the composition of the portfolio and the costs of purchasing the investment portfolio are the same. The change occurred only in later periods, when the composition of portfolio is changed on the base of model in the space of expected return and of CVaR. There were incurred also the costs associated with portfolio change that reduce the related profit of changed portfolio.

Table 2 Overall results of using the CVaR; Source: The authors

Results	Return by investment based on Table 1 without costs of change	Return by investment based on Table 1 with costs of change	Return by unchanged portfolio from 28.2.2012 without costs of change	Return by unchanged portfolio from 28.2.2012 with costs of change
27.2.2012	0.005612042	0.004412042	0.005612042	0.004412042
28.2.2012	0.002438557	0.002400725	0.002481421	0.002481421
29.2.2012	-0.013757421	-0.013761961	-0.013647052	-0.013647052
1.3.2012	0.00305953	0.002209216	0.002306798	0.002306798
2.3.2012	-0.005748302	-0.005763017	-0.013309932	-0.013309932
5.3.2012	-0.003147897	-0.003723067	-0.007140059	-0.007140059
6.3.2012	-0.029335756	-0.029419403	-0.030493842	-0.030493842
7.3.2012	0.001607007	0.001169694	0.011499984	0.011499984
8.3.2012	0.03234024	0.032203488	0.017712068	0.017712068
9.3.2012	0.00442122	0.003921558	0.00240098	0.00240098
Average	**-0.000251078**	**-0.000635072**	**-0.002257759**	**-0.002377759**

The average return is different in both cases, that results from different portfolio composition in the case of dynamically evolving or unchanged portfolio. The loss on account of costs is about 0.24% in case of model with unchanged portfolio in the space of expected return and CVaR and the loss was about 0.063%, if we are used the new information about the change in asset returns. It has the implication that the use of dynamically evolving portfolio is more appropriate as that with no change in investment weights.

4 CONCLUSION

In this paper the authors point to the portfolio selection problem. The use of static models to constitute the portfolio (in our case of the model was constructed in the space of expected yield and CVaR), which address this problem for one point in time, does not involve to take into account the cost of portfolio change, ie solving the partial problems does not guarantee the optimization of the entire process. The comparison of strategies of holding of unchanged portfolio and using optimization at each stage cannot be generalized to create a strategy for selecting a portfolio in the short period. It is therefore necessary to construct a dynamic model of portfolio selection, which would establish a way of investing in a portfolio with maximum expected net income for the entire investment.

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SOME PROPERTIES OF GRAPH FLOW PROBLEMS USED IN LOGISTICS

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Abstract. Flows in graphs are used in various problems in operations research, especially in logistics, where a flow in a network represents a flow of goods, vehicles, containers etc. There may be either one or more flows in a network. In case of multi-product flows in a graph, the flows are not independent; usually they share common capacities of arcs. In the paper, some theoretical properties of various flow problems are studied, such as: the maximum flow problem, the minimum cost problem, the minimum cost problem with fixed costs on arcs, the minimum cost problem with variable capacities of arcs, and the transshipment problem. Multi-product flow problems are studied in relation with the pickup-and-delivery problem, the skip pickup-and-delivery problem, problems with circulation of vehicles and reloading of goods. For all of the problems, issues on solvability, integrality and computational complexity are discussed.

Keywords: logistics modeling, flow problem, unimodularity

JEL Classification: C44
AMS Classification: 90C15

1 INTRODUCTION; THE GRAPH FLOW PROBLEMS

First, one-product problems are introduced. Then we turn to multi-product problems. The problem of finding a maximal one-product flow between two nodes of a graph is given as follows. A digraph (N, A) is given with two distinguished nodes: s is the *source node* and t is the *destination node*. Each arc (i, j) has a limited capacity k_{ij}. We are to find a flow $\{x_{ij}\}_{(i, j)\in A}$ through the arcs of the graph which meets *flow constraints*. The flow constraints say that the sum of inflow in each node should be equal outflow of the node (excluding source and destination nodes). For each arc (i, j), the flow x_{ij} has to be less or equal than the capacity of the arc (i, j). The goal is to find a maximal inflow in the destination node (which is, due to the flow constraints, the same as the outflow from the source node).

Assume further that total required flow between the source node and the destination node is given. The *minimal cost flow problem* consists in such a flow in the graph, which minimizes sum of costs over all arcs. The cost of an arc depends on the flow x_{ij} and the unit flow cost c_{ij}. In minimal total fixed cost problem the sum of arc fixed cost is minimized.

In the one-product transshipment problem, there are multiple sources and multiple destinations. A number a_i is associated with each node i. If the number is positive, the node is source with outflow a_i; if a_i is negative, the node i is a destination node with inflow $|a_i|$. (And if $a_i = 0$, then the node i is transit site for which the inflow equals to the outflow.)

In *multicomodity* or *multiproduct flow problems*, different commodities share common distribution capacities (networks, vehicles, containers, etc.). There are m commodities and for each node i, we have m numbers a_{hi}, where h denotes the commodity. According the sign of a_{hi}, the node is either a source, or destination or a transit site for the commodity h. The flow of the product h on the arc (i, j) is denoted x_{hij}. The sum of all commodity flows should be less or equal to the capacity of the arc (i, j).

2 MATHEMATICAL MODELS AND UNIMODULARITY OF THEIR CONSTRAINT MATRICES

An important property of an integer programming model is the integrality of its associated polyhedron. (The associated polyhedron is integral if the linear relaxation polyhedron of the model has all vertices integral.)

2.1. Definition. A matrix A is *unimodular* if it is integer and det(A) = 1 or –1.

2.2. Definition. A matrix A is *totally unimodular* if every square submatrix of A has determinant +1, –1 or 0.

2.3. Theorem. For a matrix A with entries in {+1, –1 or 0}, the following properties are equivalent:

1. A is totally unimodular.
2. The polyhedron $\{x \in \mathbb{R}^n : Ax \geq b,\ l \leq x \leq u\}$ has integer vertices for all integer-valued vectors b, l and u (n stands for dimension).
3. A' is totally unimodular.
4. For any index set J of columns of A there exists a bipartition of J (that is, a pair (J_1, J_2) such that $J_1 \bigcup J_2 = J$ & $J_1 \bigcap J_2 = \varnothing$) satisfying

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1$$

for all rows i.

2.4. Observation. A permutation of two rows (columns) of A preserves total unimodularity.

2.5. Example. Node-arc incidence matrix of a digraph is totally unimodular. Recall that the node-arc incidence matrix NA is defined as follows:

$$NA_{ij} = \begin{cases} 1, & \text{if arc } j \text{ is directed from node } i, \\ -1, & \text{if arc } j \text{ is directed towards node } i, \\ 0, & \text{otherwise.} \end{cases}$$

3 MATHEMATICAL MODELS AND THEIR MAIN FEATURES

3.1. The maximal flow problem. Input: digraph $G = (N, A)$, nodes $s, t \in N$ (source node, destination node), $k_{ij} \geq 0$ (capacity of the arc $(i, j) \in A$). A variable x_{ij} denotes the flow through the arc (i, j). We get the following model:

$$\sum_{(s,j) \in A} x_{s,j} \rightarrow \max$$

$$\sum_{(i,j) \in A} x_{ij} - \sum_{(j,k) \in A} x_{jk} = 0, \quad j \in N$$

$$0 \leq x_{ij} \leq k_{ij}, \quad (i, j) \in A$$

The matrix of the constraint is the node-arc matrix NA, which is totally unimodular, so the optimal solution X for all integer values of capacities K is integer.

3.2. The minimal costs flow problem. Input: digraph $G = (N, A)$, nodes $s, t \in N$, $k_{ij} \geq 0$ (capacity of the arc $(i, j) \in A$), c_{ij} = the cost of unit flow through the arc (i, j). T^0 = value of the required flow from the node s to the node t. A variable x_{ij} denotes the flow through the arc (i, j).

$$\sum_{(i,j)\in A} c_{ij}x_{ij} \rightarrow \min$$

$$\sum_{(s,j)\in A} x_{s,j} = T^0$$

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,k)\in A} x_{jk} = 0,\ \ j \in N$$

$$0 \le x_{ij} \le k_{ij},\ \ (i,j)\in A$$

The matrix of the constraint is node-arc matrix NA, which is totally unimodular, so the optimal solution X for all integer value of capacities K is integer.

3.3. The minimal fixed costs flow problem. Input: digraph $G = (N, A)$, nodes $s, t \in N$ (source node, destination node), $k_{ij} \ge 0$ (capacity of the arc $(i, j) \in A$), d_{ij} the fixed cost associated with the flow through the arc (i, j). T^0 = value of the required flow from the node s to the node t. A variable x_{ij} denotes the flow through the arc (i, j). The decision variable $y_{ij} \in \{0, 1\}$ fulfills $y_{ij} = 1$ iff $x_{ij} > 0$.

$$\sum_{(i,j)\in A} d_{ij}y_{ij} \rightarrow \min$$

$$\sum_{(s,j)\in A} x_{s,j} = T^0$$

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,k)\in A} x_{jk} = 0,\ \ j \in N$$

$$0 \le x_{ij} \le k_{ij}y_{ij},\ \ (i,j)\in A$$

$$y_{ij} \in \{0,1\},\ (i,j)\in A$$

The constraint matrix takes the form

$$\begin{pmatrix} NA & 0 \\ I & K \end{pmatrix},$$

where NA is the node-arc matrix and I is the unit matrix. The constraint matrix is not totally unimodular in general; it is totally unimodular only for case $K = I$.

If the variable y_{ij} is understood as a general integer, then the problem is the flow problem with variable capacity of arcs, where a capacity can be y_{ij} times the capacity k_{ij} .

3.4. The transshipment problem. Input: digraph $G = (N, A)$, $k_{ij} \ge 0$ is the capacity of the arc $(i, j) \in A$. For each node i, a number a_i is given: for a_i positive the node i is a source with capacity a_i, for a_i negative the node i is a destination with demand $|a_i|$. The unit flow cost through the arc (i, j) is c_{ij}. A variable x_{ij} denotes the flow through the arc (i, j).

$$\sum_{(i,j)\in A} c_{ij}x_{ij} \rightarrow \min$$

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = a_i,\ \ i \in N$$

$$0 \le x_{ij} \le k_{ij},\ \ (i,j)\in A$$

The matrix of the constraints is the node-arc matrix NA, which is totally unimodular, so the optimal solution X for all integer value of capacities K is integer.

3.5. Multi-commodity minimal cost flow. Input: digraph $G = (N, A)$, $k_{ij} \ge 0$ is the capacity of the arc $(i, j) \in A$. A set of commodities is denoted H. For each node i and each commodity $h \in H$, a number a_{hi} is given: for a_{hi} positive, the node i is a source with capacity a_{hi}; for a_{hi} negative, the node i is a destination with demand $|a_{hi}|$. The unit flow cost through the arc (i, j) is c_{ij}. A variable x_{hij} denotes the flow of the commodity h through the arc (i, j).

$$\sum_{(i,j)\in A}\sum_{h\in H} c_{ij}x_{hij} \quad \to \ \min$$

$$\sum_{(i,j)\in A} x_{hij} - \sum_{(j,i)\in A} x_{hji} = a_{hi}, \ i \in N, h \in H$$

$$0 \le \sum_h x_{hij} \le k_{ij}, \ \ (i,j) \in A$$

The structure of the matrix of the constraint is as follows:

$$A = \begin{pmatrix} NA & 0 & \cdots & 0 \\ 0 & NA & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & NA \\ I & I & \cdots & I \end{pmatrix}.$$

This matrix A is not totally unimodular (in general), so a solution of the problem can be non-integer for integer data.

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MAX-PLUS LINEAR SYSTEMS AT BUS LINE SYNCHRONIZATION

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Abstract
In this paper we use max-plus recurrence equations with an irreducible and regular matrix of finite size. That is associated to a model of public transport. This model focuses on computing bus line timetables and on synchronization of departures from some interchange stops at bus transportation network. We show how it is possible to minimize the waiting time for passengers who change bus lines by solution of the eigenproblem. Initial experiments with Prostějov instances of problems are presented using open source software Scicoslab.

Keywords: *max-plus algebra, linear equations, eigenproblem, discrete-event dynamic systems, bus line timetables*

AMS Classification: 15A06, 90C48

1 INTRODUCTION

Max algebra is an attractive way to describe a class of non-linear problems that appear for instance in discrete event dynamic systems [2, 3, 6]. This paper focuses on modeling bus line timetables on synchronization of departures from some interchange stops at bus transportation network [5, 8, 9]. We show how it is possible to use eigenvalues and eigenvectors of matrix to compute mean waiting time of passengers at transfer stops.. We begin with some known facts from the theory of the max-plus algebra.

2 MAX-PLUS ALGEBRA

We will use a basic notation, definitions and theorems from [1, 4]. Let N be a set of natural numbers, $\Re$ is the set of real numbers, $\varepsilon = -\infty$, $e = 0$, $\Re_{\max} = \Re \cup \{\varepsilon\}$, $\underline{n} = \{1,2,...,n\}$.

2.1 Basic Definitions

Let $a, b \in \Re_{\max}$ and let's define the operations $\oplus$ and $\otimes$ by: $a \oplus b = \max(a,b)$ and $a \otimes b = a + b$. Let $\Re_{\max}^{n\times n}$ be a set of $n \times n$ matrices with coeficients in $\Re_{\max}$. The sum of matrices $A, B \in \Re_{\max}^{n\times n}$ denoted by $A \oplus B$ is defined by $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max\{a_{ij}, b_{ij}\}$ for $i, j \in \underline{n}$. The product of matrices $A \in \Re_{\max}^{n\times l}, B \in \Re_{\max}^{l\times n}$ denoted by $A \otimes B$ is defined by $(A \otimes B)_{ij} = \bigoplus_{k=1}^{l} a_{ik} \otimes b_{kj}$ for $i, j \in \underline{n}$.

2.2 Matrices and Graphs

Let *N* be finite and non-empty set and let's consider $D \subseteq N \times N$. The pair *G* =(*N*,*D*) is called a directed graph, where *N* is the set of elements called nodes and *D* is the set of ordered pairs of nodes called arcs. A directed graph *G* =(*N*,*D*) is called a weighted graph if a weight $\omega(i, j) \in \Re$ is associated with eny arc $(i, j) \in D$. Let $A \in \Re_{\max}^{n\times n}$ be any matrix, a digraph *G*(*A*)=(*N*(*A*), *D*(*A*)) , where $N(A) = \underline{n}$ and $D(A) = \{(i, j) \in \underline{n} \times \underline{n} : a_{ji} \neq \varepsilon\}$ is called **communication graph** of *A*.

A path from node *i* to node *j* is a sequence of arcs $p = \{(i_k, j_k) \in D(A)\}_{k \in \underline{m}}$ such that $i = i_1$, $j_k = i_{k+1}$ for *k* < *m* and $j_m = j$. The path *p* consists of nodes $i = i_1, i_2, ..., i_m, j_m = j$ with length *m* denoted by $|p|_1 = m$. In the case when $i = j$ the path is called circuit. *A* circuit is called cycle if nodes i_k

and i_l are diferent for $k \neq l$. A circuit consisting of one arc is called loop. Let us denote by $P(i, j; m)$ the set of all paths form node *i* to node *j* of length $m \geq 1$ and for any arcs $(i, j) \in D(A)$ let its weight be given by a_{ji} .Then weight of path $p \in P(i, j; m)$ denoted by $|p|_\omega$ is defined by sum of the weights of all the arcs the belong to the path. The average weight of path *p* is given by $|p|_\omega / |p|_1$.

2.3 Spectral Theory

Let $A \in \Re_{\max}^{n \times n}$ be a matrix. If $\mu \in \Re_{\max}$ is a scalar and $v \in \Re_{\max}^n$ is a vector of at least one finite element such that: $A \otimes v = \mu \otimes v$ then, μ is called an **eigenvalue** and *v* an **eigenvector**. Note that the eigenvalue can be equal to ε and is not necessarily unique. Eigenvectors are certainly not unique by definition.

Let *C*(*A*) denote the set of all cycles in *G*(*A*) and write: $\lambda(A) = \max_{p \in C(A)} \frac{|p|_\omega}{|p|_1}$ for the maximal everage cycle weight. Note that since *C*(*A*) is a finite set, the maximum is reached in case $C(A) = \emptyset$ define $\lambda = \varepsilon$.

Theorem 1 (Bacelli at al. [1]) Let $A \in \Re_{\max}^{n \times n}$ be irreducible. Then there exists one and only one finite eigenvalue (with possible several eigenvectors). This eigenvalue is equal to the maximal average weight of cycles in *G*(*A*).

Cuninghame-Green (1960) showed (cited in [4]) that λ (*A*) is an optimal solution of the linear program

$$\lambda \to \min \qquad (1)$$

$$\lambda + x_i - x_j \geq a_{ij} \qquad \forall (j,i) \in D(A), \qquad (2)$$

$$x_i \geq 0 \qquad \forall i \in N(A). \qquad (3)$$

An efficient way of evolution λ (*A*) is Karp's algorithm of complexity $O(n^3)$ or almost in linear time using Howard's algorithm. Because λ (*A*) is the optimal value of linear program (1-3) is valided following fact.

Theorem 2 (Cechlárová [4]) Let $A \in \Re_{\max}^{n \times n}$ be a matrix. Than inequality

$$A \otimes x \leq \mu \otimes x \qquad (4)$$

is solvable if and only if $\mu \geq \lambda$ (*A*).

We say that a real number μ and vector *x* from (4) are solutions of an equation, they are called an **approximate eigenvalue** and an **approximate eigenvector** of a matrix $A \in \Re_{\max}^{n \times n}$.

3 SYNCHRONIZATION OF DEPARTURES AT BUS LINE TIMETABLES

In this paper we make use of two interpretations of eigenproblem:

- If the public transport system is performed in cycles and consists of jobs with matrix describing duration operation *A* has been started according to some eigenvector *x* of the matrix, then it will move forward in regular steps. The time elapsed between the consecutive starts of all jobs wil be equal to λ (*A*).

- Let's have a bus schedule for the system that requires that time interval between two consecutive jobs should not exceed a certain value μ. In this case idle times of systems can be used for optimization waiting time of passengers.

3.1 Max-plus model

We will compose two models of synchronized timetables of bus network. Both models use following parameters:

- S set of stops of bus network,
- S_C set of transfer stops of bus network; $\emptyset \neq S_C \subset S$,
- q number of lines in the network,
- n number of line directions in the network,
- i index of line directions; $i \in \underline{n}$,
- t_i traveling time in the direction; $i \in \underline{n}$,
- α_i first stop of the line direction i; $\alpha_i \in SS$,
- β_i last stop of the line direction i; $\beta_i \in SS$,
- o_q turnaround time of the line q,
- m_q number of buses on the line q,

and variables:

- $x_i(k)$ k^{th} bus synchronized departure time in direction i of timetable,
- λ length of the mean period between departure times of timetable.

The model solves following synchronization of departure times of timetable on crossing stops of bus network (BSP - Basic Synchronization Problem): Given bus network with the set of bus lines, the sets of stops and transfer stops of bus lines. We know traveling time on line directions and turnaround time of the lines. We suppose that every line is covered by given number of buses and buses do not change lines. The goal is to find the synchronized departure time of timetables when a change of line on transfer stops are required. The objective function, the maximal difference between departure times of same directions, is minimized.

Now we can define matrix $A \in \Re_{\max}^{n \times n}$ where finite elements a_{ij} give the time between departute time from stop a_j and direction j and arrival time to stop b_i of direction i.More details can be found in Turek's disertation work [9]. Because vector $x(k) = (x_1(k), x_2(k), ..., x_n(k))$ denotes the time in wich all buses started for the k^{th} cycle then if all buses wait for all preceding jobs to finish their operations, the earliest possible starting time at $(k+1)^{th}$ cycle are expressed by vector $x(k+1)$, where

$$x_i(k+1) = \max\{a_{i1} + x_1(k), a_{i2} + x_2(k), \ldots, a_{in} + x_n(k)\}, \tag{6}$$

and can be expressed by vector equation ower max-plus algebra of the form

$$x(k+1) = A \otimes x(k). \tag{7}$$

If a initial condition

$$x(0) = x, \tag{8}$$

is given, the whole future evolution of (7) is determined. When x is an eigenvector and λ is an eigenvalue of matrix A then in general

$$x(k+1) = A^k \otimes x(0) = \lambda^k \otimes x. \tag{9}$$

So the mean waiting time at crossing stops of with a unit intensity of passenger‘s flow has in our system the value $\omega(A)$, where

$$\omega(A) = \frac{\sum_{i\in \underline{n}} \sum_{j\in \underline{n}: a_{ij} \rangle \varepsilon : \beta_j = \alpha_i \in S_C} \left(\lambda + x_i - x_j - a_{ij}\right)}{\sum_{i\in \underline{n}} \sum_{j\in \underline{n}: a_{ij} \rangle \varepsilon : \beta_j = \alpha_i \in S_C} 1}. \tag{10}$$

It is possible to show that the matrix (5) of system A is irreducible and also from Theorem 1 there exists only one λ which can by computed as the maximal everage cycle weight $\lambda\,(A)$.

In case that the eigenvalue or eigenvector are fractional then is inappropriate for real timetable. Than we can solve

$$A \otimes y = \mu \otimes y, \tag{11}$$

where y is **integer approximate eigenvector** for given approximate eigenvalue μ. Corresponding approximate mean waiting time at transfer stops can be calculated similarly as in form (10). Open question stays how to find effectively an integer approximative eigevector.

3.2 Prostějov experiments

Scicoslab [7] is a free environment for scientific computation similar in many respects to Matlab/Simulink, providing Matlab functionalities. The maxplus toolbox is included in ScicosLab it had been used in our computation experiments.

The modeling issues will be ilustrated by a simple fragment of bus network with stops $S = \{A, B, C, D, E\}$ in city Prostějov for two bus lines 1 and 2 are given in Figure 1. Lines are servised by one bus only.

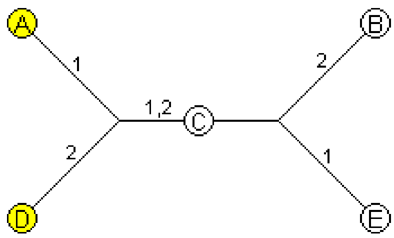

Figure 1: Ilustrative scheme ispired by real Prostějov bus network

Corresponding matrix A synchronize change a lines for a ow passenger between lines 1 and 2 at crossing stop C.

$$A = \begin{array}{c} \\ (A,C) \\ (C,A) \\ (B,C) \\ (C,B) \\ (D,C) \\ (C,D) \\ (E,C) \\ (C,E) \end{array} \begin{array}{cccccccc} (A,C) & (C,A) & (B,C) & (C,B) & (D,C) & (C,D) & (E,C) & (C,E) \\ 54 & 15 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 39 & \varepsilon & 17 & \varepsilon & 6 & \varepsilon & 12 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 21 & 27 & \varepsilon & \varepsilon & \varepsilon \\ 10 & \varepsilon & \varepsilon & \varepsilon & 6 & \varepsilon & 12 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 54 & 10 & \varepsilon & \varepsilon \\ 10 & \varepsilon & 17 & \varepsilon & 44 & \varepsilon & 12 & \varepsilon \\ 27 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 17 \\ 10 & \varepsilon & 17 & \varepsilon & 6 & \varepsilon & \varepsilon & \varepsilon \end{array} \tag{12}$$

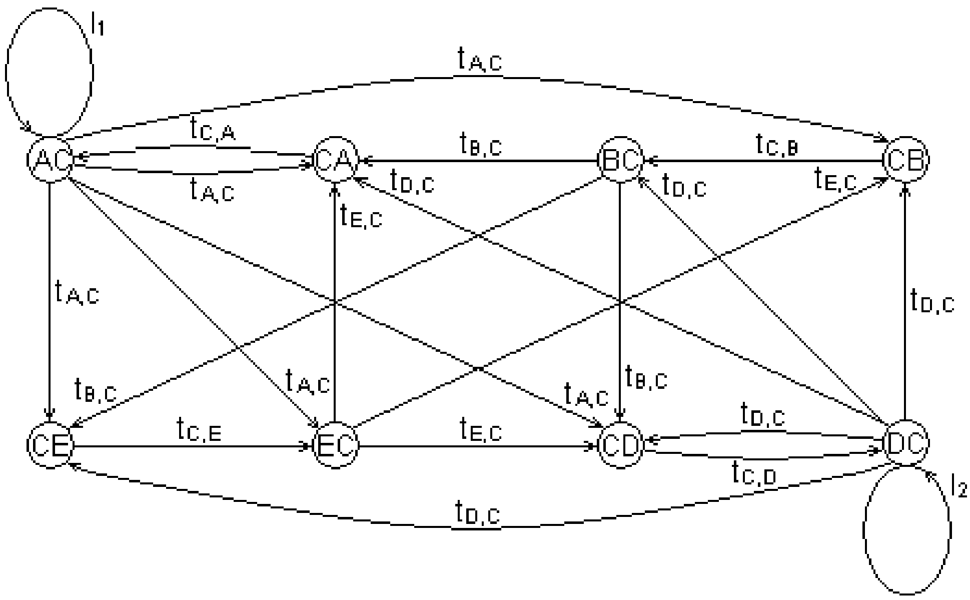

Figure 2: Communication graph *G*(*A*)

In the communication graph *G*(*A*) on figure 2 we have a set of vertices $N(A)=\{(A,C),(C,A),(B,C),(C,B),(D,C),(C,D),(E,C),(C,E)\}$ noted for easier orientation. As we can see, the graph *G*(*A*) is strongly connected, between every difstict vertex exists oriented path. So the matrix *A* is isreducible and or instance have only one eigenvalue $\mu = 54$ The eigenvector $x = (54,39,54,44,27,10,27,10)$. Then the mean waiting time at crossing stops $\omega(A) = 4{,}9$

4 CONCLUSION

This paper introduced model of synchronization of departures at bus line network based on spectral theory of max-plus algebra. Computation experiments by using open source software scicoslab on Prostějov bus network shows that this aproach is applicable on real bus network and generates interesting open problems.

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A TWO PHASE APPROACH TO REVERSE LOGISTICS NETWORK DESIGN

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Abstract
This paper presents modeling approaches, which can be used to establish a reverse logistics network for EOL products. We proposed a two phase mixed integer linear program model for defining optimum locations for collection points, transfer stations and recycling facilities. In the first stage, proposed model finds a minimum number of collection point's locations depending on the distance from end users to collection points. The second model's phase defines optimal locations of a transfer and recycling facilities while minimizes the reverse logistics costs. The proposed modeling approach was tested on the Belgrade city area.

Keywords: *logistics network design, recycling, facility location*

JEL Classification: C61
AMS Classification: 90C11

1 INTRODUCTION

The increase in municipal waste generation in OECD countries was about 58% from 1980 to 2000, and 4.6% between 2000 and 2005 [OECD, 2008]. In 2030, the non-OECD area is expected to produce about 70% of the world's municipal waste, mainly due to rising incomes, rapid urbanization, and technical and economic development. So, the problem facing nowadays society is finding options to recover discarded products, in order to prevent waste generation and conserve natural resources. Introduction of new management policies had strong impact on waste management practices like recycling. Recycling rate in EU15 in 2005 was about 41%–up from 22% in 1995 [OECD, 2008]. It is expected that the recycling rate may increase even more rapidly, due to the emerging recognition of the economic and environmental benefits of recycling, compared to other waste management option e.g. incineration.

However, recycling as one solution for the waste management problem requires appropriate logistics network structure, as well as adequate realization of logistics activities in such a network. In fact, most of existing logistics systems aren't designed to support reverse logistics activities, because the presence of recovery options, such recycling, introduces some new characteristics of logistics systems. So, it is necessary to redesign an existing logistics network to handle returned products, or set up a completely new reverse logistics network.

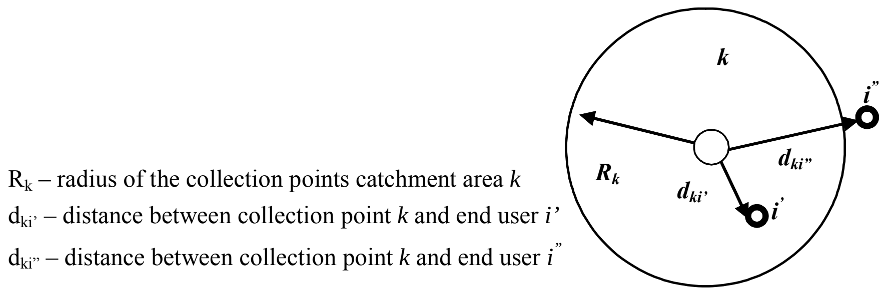
From here, the main intention of this research was to analyze the modeling approaches, which can be used to establish a reverse logistics network for EOL products. We proposed a two phase mixed integer linear program model for determining optimum locations for collection points, transfer stations and recycling facilities. In the first stage, the proposed model finds a minimum number of initial collection point's locations depending on the distance from end users to collection points. In order to model the influence of distance between end users and collection points on the optimal locations of facilities to be located, we introduce the collection point's catchment area. The second phase of the model defines optimal locations of transfer and recycling facilities while minimizes the reverse logistics costs. The proposed modeling approach was tested on the case of Belgrade city area.

With this objective, the remaining part of the paper is organized as follows. Section two describes analyzed problem, while the next section presents the mathematical formulation. The numerical results of the modeling approaches for the case of Belgrade city are shown in section four. Finally, some concluding remarks are made.

2 PROBLEM DESCRIPTION

In order to maximize the value of EOL products, it is necessary to establish such a logistical structures to manage the reverse flow of goods in an optimal way. In order to make EOL products available for recycling, the first step is an effective collection from the EOL generators. The aim is, certainly, to provide larger quantities of these products because they represent an input in the recycling process. Many studies demonstrated that the decision to participate in recycling activities is influenced by the provision of waste collection bins and easily accessible collection sites (González-Torre and Adenso-Díaz, 2005; Domina and Koch 2002; Garcés et al, 2002). Hence, an appropriate collection site can be selected by taking into consideration the geographic location, the ease and convenience to consumers, and the population distribution.

Figure 1 Collection point's catchment area

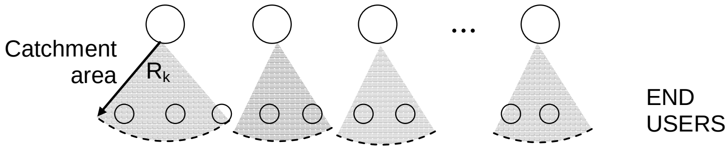


So, in order to model the influence of distance between users and collection points on the collecting of EOL products, we introduce the collection point's catchment area (Figure 1). The catchment area models the influence of distance between end users and collection points, in the sense that for all end users and collection points, collection service may exist only when end users are within the certain (reasonable) distance from a collection point *k*. Therefore, catchment area denotes the area within the circle of certain predefined radius from the drop-off location. That is, any arbitrary end user can be allocated to the collection point only if it is located within the collection point's catchment area.

3 MATHEMATHICAL FORMULATION

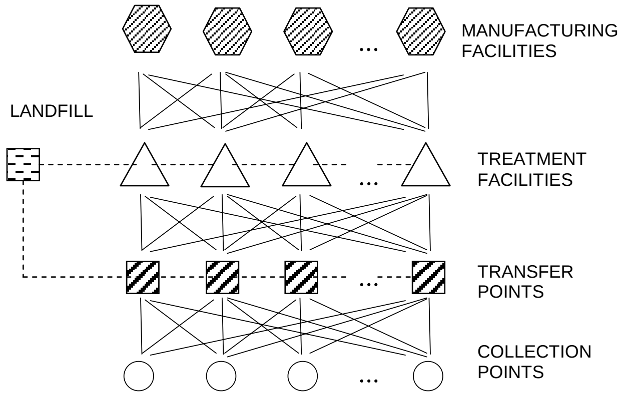
Planning of reverse logistics networks for large urban centers becomes very complicated. For example, the city of Belgrade to the last available census (2002) has 1.576.124 inhabitants and covers an area of 3205 km^2. If entire network was optimized at once, then this problem becomes practically unsolvable for the optimal case. For that reason, this paper proposes a two-phase approach to the formulation of the problem. The first phase of the reverse logistics network design refers to finding locations of collection points dependent on the radius of collection point catchment area (Figure 2) .

Figure 2 First level of reverse logistics network



The second level of reverse logistics network refers to determining optimal locations for transfer stations and recycling centers (**Figure 3**)

Figure 3 Second level of reverse logistics network



For the first phase of reverse logistics network design the following mathematical formulation is proposed:

$$\text{Min} \sum_k \sum_p Y_{k+1} F_{k+1} \tag{1}$$

s.t.

$$\sum_k X_{ik} = 1 \ldots \forall i \tag{2}$$

$$X_{ik} \le Y_k \ldots, \forall i, k, \tag{3}$$

$$(d_{ik} - R) X_{ik} \le 0, \ldots \forall i, k, \tag{4}$$

$$Y_{k+1} \in \{0,1\},\ X_{ik} \in \{0,1\} \tag{5}$$

Where

Sets

$I = \{1, \ldots N_i\}$	End users location
$K = \{1, \ldots N_k\}$	Potential locations of collection point

Parameters

d_{ik}	distance between end user *i* to collection point k , $i \in I$, $k \in K$
R_k	radius of the catchment area for collection point k, $k \in K$
F_k	Fixed costs of opening location *k*

Variables

X_{ik}	Binary variable that defines EOL flow allocated from end user *i* to collection point *k*.
Y_{k+1}	Binary variable, Y_k=1 if collection point *k* is opened, otherwise Y_{k+1} =0, $k \in K$

The last node Y_{k+1} represents a dummy site with infinite cost which prevents infeasibility in the solution procedure. A dummy node was included to collect product flows from end user with a distance greater than R_k from any opened collection pont *k*. The objective function (1) minimizes the cost of opening collection points. The first set of constraints (Equation 2) ensures that all EOL products currently located at end user are transferred to collection points (guaranted that all products for recycling, currently at end users, are delivered to collection sites). Constraints 2 represent collection points opening constraints, stating that EOL products from the end user *i* can be transferred to collection point *k* only if it is opened. Set of constraints 3 represents the catchment area of the collection point *k* and allow allocation of end users to collection points only when the distance between end user and the collection point is within the predefined radius of the catchment area. Finally, last constraints define binary nature of variables.

For the next level of reverse logistics network, i.e. determining optimal locations for transfer stations and recycling centers, the following mathematical formulation is proposed:

$$\text{Min} \sum_k \sum_l \sum_p C_{pkl} X_{pkl} + \sum_l \sum_j \sum_p C_{plj} X_{plj} + \sum_l \sum_p C_{plj+1} X_{plj+1} + \sum_p \sum_j Y_{jp} F_{jp} + \sum_p \sum_l Y_{lp} F_{lp} + \sum_p \sum_k C_{pkD} X_{pkD} \quad (6)$$

$$\sum_l \sum_p X_{pkl} + X_{pkD} = q_{kp} \ldots \forall k, \forall p \quad (7)$$

$$\sum_p \sum_k X_{pkl} - \sum_p \sum_j X_{pkl} - X_{pkl+1} = 0 \ldots, \forall l, \forall p \quad (8)$$

$$X_{pkl+1} = \alpha_p \sum_p \sum_j X_{pkl} \ldots, \forall l, \forall p \quad (9)$$

$$\sum_p \sum_k X_{pkl} \le Y_{lp} G_{lp} \ldots \forall l, \forall p \quad (10)$$

$$\sum_p \sum_l X_{plj} \le Y_{jp} G_{jp} \ldots \forall j, \forall p \quad (11)$$

$$X_{pkl} \le Y_{lp} G_{lp} \ldots \forall l, \forall p \quad (12)$$

$$X_{plj} \le Y_{jp} G_{jp} \ldots \forall j, \forall p \quad (13)$$

$$X_{pkl}, X_{plj}, X_{plj+1}, X_{D\,p} \quad > 0 \quad (14)$$

$$Y_{lp}, Y_{jp} \in \{0,1\} \quad (15)$$

Where

Sets

$K = \{1,...N_k\}$	Locations of collection point
$L = \{1,...N_l\}$	Potential locations for transfer stations
$J = \{1,...N_{j+1}\}$	Potential locations for recycling centers, where *j+1* represents landfill location
$P = \{1,...N_p\}$	Types of products

Parameters

p	Product type, $p \in P$
α_p	minimal disposal fraction of product type *p*, $p \in P$
G_l	capacity of transfer point *l*, $l \in L$
G_j	capacity of recycling facility *j*, $j \in J$
C_{pkl}	transportation costs of transporting EOL product *p* from collection point *k* to transfer facility *l*, $p \in P, k \in K, l \in L$
C_{plj}	transportation costs of transporting EOL product *p* from transfer point *l* to treatment facility *j*, $p \in P, l \in L, j \in J$
C_{plj+1}	transportation costs of transporting EOL product *p* from transfer point *l* to landfill site *j+1*, $p \in P, l \in L, j+1 \in J$
F_{lp}	Fixed costs of opening transfer station for product *p* on location *l*
F_{jp}	Fixed costs of opening recycling center for product *p* on location *j*

Variables

X_{pkl}	fraction of a product *p* transported from collection site *k* to transfer point *l*, $p \in P$, $k \in K$, $l \in L$. Variable X_{pkD} represents a dummy site with infinite cost and infinite capacity which prevents infeasibility in the solution procedure
X_{plj}	fraction of a product *p* transported from transfer point *l* to treatment facility *j*, $p \in P, l \in L, j \in J$
X_{plj+1}	fraction of a product *p* transported from treatment facility *j* to manufacturing facility *s*, $p \in P, j \in J, s \in S$
Y_{lp}	Binary variable, Y_{lp}=1 if transfer point *l* is opened, otherwise Y_{lp} =0,
Y_{jp}	Binary variable, Y_{jp}=1 if recycling center *j* is opened, otherwise Y_{jp} =0,

Objective function (6) minimizes total transportation costs of transporting products from collection points, via transfer stations to treatment facilities, as well cost of opening those facilities. All products currently located at collection points must be transferred to transfer stations, which is presented with fist constraint. Constraints (8)-(9) are flow conservation constraints, while constraint (9) models minimum disposal fraction form transfer point level. Constraints (10) to (13) are capacity and opening constraints, while constraints (14) and (15) define binary and continuous nature of the variables.

4 NUMERICAL EXAMPLE

The proposed models are tested on the Belgrade city example. In general, the following two types of residential areas in Belgrade city may be distinguished: low density zones, with lodgings like houses, and high density zones, with lodgings on many levels (sky scrapers and similar buildings). In order to optimize reverse logistics network for entire Belgrade city, it was necessary, for both types of housing, to take the typical representatives of local communities (LC) as the smallest areas of urban settlements in Serbia. And then determine the optimal number of locations for the collection of recyclables. Latter, based on certain parameters (population, area, population density) it was necessary to determine the number of locations for all the other local communities (total 327).

The first stage of designing the logistics network is related to the determination of number of locations for the collection of recyclables based on the distance from the end users and potential locations for the collection. Model 1 is solved using the IBM ILOG CPLEX 12.2 software and the model results for different values of radius catchment area for representative LC are presented Table 2. The distances from the end users to potential locations for the collection are determined using the GIS software, while the fixed costs of opening location were the same for all possible locations and amounts 150 euros.

Table 1 The results obtained by solving the first model for a typical representatives of LC

	Objective function value	Radius of collection point catchments area (m)	Number of opened collection points
LC Stari aerodrom (Municipality New Belgrade)	4350	50	29
	2100	100	14
	1200	150	8
	750	200	5
	450	250	3
LC David Pajić (Municipality Voždovac)	34500	50	230
	16200	100	108
	6900	150	46
	4650	200	31
	2400	250	16

After determining the number of locations for collection points, as has been said, it was necessary to categorize the remaining LC according to the type of housing, and then to specify the number of collection points within those LCs The number of collection locations for the remaining 325 LCs is estimated on the base of its. After that, the next level of reverse logistics network is optimized. Potential locations for transfer stations and recycling centers are taken from the Urban Plan of Belgrade for the year 2021. For the calculation of the distance of potential sites to LC centers, GIS was used, whereas for the calculation of distances within the LC, and the distance between the individual collection sites within a particular LC is used the following approximate formula:

$$D(TSP) \approx 0.765\sqrt{nA} \qquad (16)$$

Where A represents an area of LC, n represents a number of collection points that need to be served, and the 0.765 coefficient valid for the Euclidean distance. Capacity of recycling centers and transfer stations, and fixed costs are generated randomly in the interval [0, 100 000] and [0, 10 000] respectively. The waste quantities are estimated according to generated quantities in Belgrade (http://www.sepa.gov.rs/download/otpad.pdf). Percentage of three types of recyclables that ends in a landfill are 0.10, 0.05 and 0.2. The results obtained by solving the second mathematical model are presented in Table 2.

Table 2 The results obtained by solving second mathematical model.

Radius of collection point's catchment area (R_k)	Objective function value	Product type	Number of served LC	Opened transfer stations	Opened recycling centers
50m	58894799.3458	glass	196	Rakovica	Rakovica
		PET bottles	327	Zemun1	Rakovica
		cans	327	Rakovica	Grocka1
100m	57475143.6199	glass	209	Rakovica	Rakovica
		PET bottles	327	Zemun1	Rakovica
		cans	327	Rakovica	Grocka1
150m	56426907.8534	glass	222	Rakovica	Rakovica
		PET bottles	327	Zemun1	Rakovica
		cans	327	Rakovica	Grocka1
200m	56027783.5256	glass	224	Rakovica	Rakovica
		PET bottles	327	Zemun1	Rakovica
		cans	327	Rakovica	Grocka1
250m	55543988.4964	glass	232	Rakovica	Rakovica
		PET bottles	327	Zemun1	Rakovica
		cans	327	Rakovica	Grocka1

The value of the objective function is increasing with decreasing of radius catchment area, which is quite understandable given the fact that increasing radius of the catchment area reduces the number of collection points that need to be served and objective function tends to minimization of total system costs. As for the number of served LC (327 in total), in the case of glass containers, table 3 shows that not all LC are served. This result is a consequence of the capacity of transfer stations and recycling centers that are generated in a random way (glass packaging is much heavier than PET and metal containers).

5 CONCLUSION

The integration of reverse flow of products into existing or planned logistic systems has become a very important issue in the last decade of the twentieth century, both from the theoretical and practical point. The problem of determining the location of collection points, transfer stations and recycling centers have an important role because high logistics costs**.** This paper presents modeling approaches, which can be used to establish a reverse logistics network for EOL products. We proposed a two phase mixed integer linear program to defining optimum locations for collection points, transfer stations and recycling facilities. In the first stage, the proposed model finds a minimum number of collection point's locations depending on the distance from end users to collection points. The second model defines optimal locations of transfer and recycling facilities while minimizes the reverse logistics costs. The main contribution of this paper is in testing the model which respects reverse logistics system on an real example, and in analyzing the impact of collection point's catchment area. Of course, proposed approach should be understood only as beginning of the more thoroughly research which is just opened, where one possible extension may be related to analyze possibilities for defining collection points catchment area as a function of socio demographic and other relevant characteristics of potential users.

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SOLVING VEHICLE ROUTING PROBLEM USING BEE COLONY OPTIMIZATION ALGORITHM

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Abstract

This article is focused on solving the vehicle routing problem by applying bee colony optimization algorithm. Vehicle routing problem is more complex variant of traveling salesman problem, because we have to consider many constraints e.g. vehicle capacity, time windows, diversity of fleet. When solving this problem we also have to solve the knapsack problem, because each vehicle has its defined capacity that has to be utilized optimally. Bee colony optimization algorithm belongs to metaheuristics so we can apply it to solve NP-hard problems. To demonstrate algorithms' ability to solve this kind of problems we created C# program that utilizes bees behaviour during nectar collection. Bees are divided into three groups, active bee, inactive bee and scout. Active bee scans space close to the food source (in our case, food source is city), scout randomly searches through whole space of solutions, and inactive bee waits for waggle dance of active bee and scout. This waggle dance is used for information exchange. Thanks to scouts the hive can leave local extreme, because of their random search of whole space of solutions.

Keywords: *BCO, ABC, VRP, metaheuristics*

JEL Classification: C44
AMS Classification: 90C15

INTRODUCTION

Vehicle routing problem (VRP) belongs to the most studied combinatorial optimization problems. When solving this problem we need to find the optimal route diagram, which will follow all vehicles to meet the needs of all customers with minimal transportation costs. Since 1959 when this issue was firstly described in work of Dantzig and Ramser [2], many studies have been devoted to the exact and approximate solutions to this problem and its variations.
The most popular variations include, for example capacitated vehicle routing problem (CVRP) with homogenous fleet where the only limitation is the amount of goods each vehicle can transport, vehicle routing problem with time windows (VRPTW) where each customer has defined time period when he needs to be visited. Nowadays much attention is given to more complex VRP variants, also called "rich" VRP, which describe real distribution problems more accurately. Their characteristics are use of more depots, vehicle fleet diversity, operational problems, loading constraints an others.

1 BEE COLONY ALGORITHM

Bee colony algorithm (BCO) sometimes denoted as Artificial Bee Colony (ABC) simulates behaviour of bees during nectar collection. Because flying is for bees very demanding and exhausting they tend to optimize food collection process to maximize the amount of collected nectar and minimize the distance travelled. The main idea of BCO is to create multi-agent system (colony of artificial bees) that is able to solve complex combinatorial optimization problems.
Bees are highly organized social insects. The survival of the entire colony depends on each individual bee. To ensure the existence of colony, bees are using a systematic segregation of duties among them. Bees perform various tasks such as foraging, reproduction, taking care

of young, patrolling, cleaning and building the hive. From these tasks the most important is foraging, because bees must be able to ensure a continuous supply of food for the colony.
The behavior of bees during foraging was mysterious for many years until Karl von Frisch decrypted the language embedded in bee waggle dance. Bees use this dance to communicate with each other. Suppose that a bee has found a rich food source. After its return to the hive, begins to dance in the form of number eight. Through this informative dance bee is able to inform its mates about direction and distance to the newly discovered food source and thus it can attract more bees.

1.1 BCO algorithm implementation

We decided to solve vehicle routing problem with capacity constrain and homogenous fleet of vehicles. Next we will describe the implementation of BCO algorithm.
Bees are divided into three groups, each performing specific task:

- Active bee (Foraging): generates neighborhood solutions to solutions that are currently in its memory,
- Inactive bee (Observing): waits in the hive for waggle dance of foraging and active bees. Observing bee can follow other bee based on its waggle dance and defined probability. Foraging bees that was unable to improve its solution for a given number of iterations, becomes Observing bee and at the same time random Observing bee becomes Active bee.
- Explorer (Scouting): generates random solutions. This is important for the algorithm to be able to leave local extreme.

Each bee has its solution in memory. This solution is always a complete solution to the given problem that consists of set of routes for each vehicle. Active bee uses GetNextSolution() method to generate neighborhood solution that consists from two steps SwapCitiesInOneRoute() and SwapCitiesAmongRoutes(), these are described in Figure 1 and Figure 2. Explorers generate random solutions completely independent of the current in its memory. If Active or Scout bee finds better solution than was its current one, it gets updated. Bee then call DoWaggleDance() method and also compares its new solution to the global best solution. If bee has found a better solution than was the global one, it gets updated.

Figure 1

```
ForagingBee.ProcessState
{
        Probability = random(0,1);
        NewSolution = Bee.GetNextSolution();            //bee found better solution that its current one
        if(NewSolution.Length() < Bee.Solution.Length())
        {
                if(Probability < ProbabilityMistake)    // bee makes mistake
                {                                        //ignores better solution
                        NumberOfVisits++;
                }
                else                                     // no mistake
                {
                        Bee.Solution = NewSolution;     // updates current solution with better one

                        DoWaggleDance();                 // bee performs waggle dance
                }
        }                                                // bee did not find better solution that its current one
        else
        {
                if(Probability < ProbabilityMistake)    // bee makes mistake
                {                                        // updates current solution with worse one
                Bee.Solution = NewSolution;
                }
                else                                     // bez chyby
                {                                        //ignores worse solution
                        NumberOfVisits++;
```

```
        }
    }                           //bee was unable to improve its solution for given number of iterations
    if(NumberOfVisits == MaxVisits)
    {                           //deactivate bee
        Bee.State = observing;
                                //activate observing bee
        ObservingRoom.RandomBee.State = foraging;
    }                           //bee found better solution than global
    if(Bee.Solution.Length < GlobalSolution.Length)
    {
        GlobalSolution = Bee.Solution;
    }
}
```

Figure 2

```
ScoutingBee.ProcessState
{
    NewSolution = RandomSolution();                 //bee found better solution
    if(NewSolution.Length < Bee.Solution)
    {                                               // updates current solution with better one
        Bee.Solution = NewSolution
                                                    // bee found better solution than global
            if(Bee.Solution.Length < GlobalSolution.Length)
        {
            GlobalSolution = Bee.Solution;
        }
                                                    // bee performs waggle dance
        DoWaggleDance();
    }
}
```

As you can see Foraging bee makes a mistake with probability of `ProbabilityMistake` that in our case was set to 0,01 (bee makes mistake with probability of 1%). Scouting bee never makes mistake.

2 EXPERIMENTAL RESULTS

Our version of BCO was coded in C# as a parallel application. Bellow are results for data sets that can be found at http://www.branchandcut.org/VRP/data

Table 1

Data set	Optimal Value	No. Cities	No. Vehicles	Tight.	Computed value	Deviation	Execution time
E-n13-k4.vrp	247	13	4	76%	247	0,00%	0:00:00:35
					247	0,00%	0:00:00:36
					250	1,21%	0:00:01
					251	1,62%	0:00:04
E-n31-k7.vrp	379	31	7	92%	387	2,11%	00:05:49
					390	2,90%	00:09:34
					395	4,22%	00:05:07
P-n55-k7.vrp	568	55	7	88%	576	1,41%	0:19:12
					580	2,11%	0:02:40
					581	2,29%	0:19:49
					582	2,46%	0:06:29

P-n101-k4.vrp	681	101	4	91%	718	5,43%	0:43:31
					724	6,31%	0:53:04
					729	7,05%	0:49:00
					742	8,96%	0:33:11
M-n151-k12.vrp	1053	151	12	93%	1123	6,65%	0:48:50
					1129	7,22%	0:44:37
					1149	9,12%	0:49:27
					1152	9,40%	0:57:01
M-n200-k17.vrp	1373	200	17	94%	1475	7,43%	1:25:44
					1503	9,47%	0:53:44
					1519	10,63%	2:36:39
G-n262-k25.vrp	6119	262	25	97%	6970	13,91%	6:10:41
					7231	18,17%	3:23:49
					7335	19,87%	3:49:15

As Table 1 shows we were able to get very good results (with deviation near 5%) in reasonable amount of time for data set with approximately 100 cities. These results were achieved on PC with AMD Athlon II X2 245 (2,9GHz) processor, 2GB RAM memory, and Windows 7 Enterprise 32bit.

3 CONCLUSION

The experimental results confirm that the BCO algorithm is able to solve vehicle routing problems in reasonable amount of time. Currently we are working on improving achieved results by implementing 2-opt heuristics to GetNextSolution() method. We expect that this optimization will reduce computing time.

This application can also be used to solve Travelling Salesman Problem (TSP) by modifying number of vehicles to 1 and capacity of the vehicle to a big number, that it can satisfy the total demand of all cities.

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GOAL PROGRAMMING – HISTORY AND PRESENT OF ITS APPLICATION

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Abstract
Goal programming, a modification and extension of linear programming, was originally formulated by Charnes, Cooper and Ferguson in 1955 and named in the Volume I of their work Management Models and Industrial Applications of Linear Programming (1961). Since that time many methods based on goal programming approach and their applications were introduced. The aim of this paper is to present some of the important applications from over the past 50 years. Also the current trends are mentioned.

Keywords: *goal programming, allocation problem, timetabling*

JEL Classification: C44
AMS Classification: 90C29, 90C90

1 INTRODUCTION

Goal programming is both a modification and extension of linear programming. It is a widely used approach for solving not only multi-objective decision problems. Goal programming was first formulated in 1955 by Charnes, Cooper and Ferguson [2] but named a few years later in 1961 in work of Charnes and Cooper Management Models and Industrial Applications of Linear Programming [3]. Further development ([5], [6], [8]) leads goal programming approach to become a widely used technique for solving decision problems. Due to quite easy formulation of the goal programme and good understanding its methodology by decision makers, many papers and books dealing with goal programming applications appeared since that time.
The aim of this paper is to present some of the important applications of goal programming from over the 50 years of its existence.

2 GOAL PROGRAMMING[1]

Goal programming (GP) is based on assumption that the main decision-maker objective is to satisfy his or her goals. Another basic principle that is used in GP is optimisation; decision-maker wants to choose the best solution from all the possible solutions. In this case we speak about Pareto-optimal solution, which means we cannot improve value of one of the criteria without making other criterion worse. We can find elements of optimisation in GP in case we try to get as near as possible to optimistically set goals. Last but not least, GP uses the balance principle, which is based on minimisation of maximal deviation from set goals.
Let us assume that the problem has generally K goals that we denoted with indices $k = 1, 2 \dots, K$. Then we define n variables $\mathbf{x} = (x_1, x_2\dots, x_n)$. These variables are factors that determine the solution and the decision-maker can influence them. Then a general goal programme can be formulated as follows:
Minimise

$$z = f(\mathbf{d}^-, \mathbf{d}^+), \tag{1.1}$$

subject to

$$\mathbf{x} \in X, \tag{1.2}$$

$$f_k(\mathbf{x}) + d_k^- - d_k^+ = g_k, \quad k = 1, 2, \dots, K, \tag{1.3}$$

$$d_k^-, d_k^+ \geq 0, \quad k = 1, 2, \dots, K, \tag{1.4}$$

[1] This chapter is based on [7].

where (1.1) is a general achievement function, X is the set of feasible solution satisfying all of the constraints including non-negativity constraints, $f_k(\mathbf{x})$ is an objective function that represents the k-th criterion, g_k is the k-th goal, the decision maker wants to meet, d_k^- is negative deviation from the k-th goal, which represents the underachievement of the k-th goal, d_k^+ is positive deviation from the k-th goal, which shows the overachievement of the k-th goal. Both deviations are non-negative and only one of them can be positive. Therefore the model should include also the constraint $d_k^- \cdot d_k^+ = 0$, $k = 1, 2, \ldots, K$, which ensure that one of the deviations would be zero. Nevertheless this constraint is fulfilled automatically due to the nature of achievement function minimisation.

3 APPLICATIONS OF GOAL PROGRAMMING

3.1 Applications of Goal Programming to Education

In 1976 published Van Dusseldorp et al. [12] three applications of GP to educational problems – problem of scheduling instruction (differences of GP and LP model please find in [9]), problem of busing (transportation of pupils from a certain area to schools) and problem of job factor compensation (determining job factor compensation for supervisory personnel under collective bargaining).

3.2 Public Health Resource Allocation

With the Special Supplement Food Program for Women, Infants, and Children (WIC) deal Tingley a Liebman [11]. The aim of this program is to "provide nutritious food supplements and nutrition education to low income pregnant and breastfeeding women, and infants and children through age five identified as being at nutritional risk" [11, p. 279]. The support id distributed through local agencies. WIC are divided into six priority subgroups. The model serves to allocate the budget for next year. The aim of the developed model is to minimize the deviations from given numbers of additional supported persons at each agency and in each priority subgroup without exceeding the given budget. In the paper [11] the model is applied on the county of Indiana WIC program.

3.3 Allocating Children to schools

Sutcliffe, Boardman and Cheshire [10] developed a model for allocating children to secondary schools in Reading (Berkshire, UK; about 60 km west of London) using goal programming approach. Their model had six goals, which were to minimise deviations from the average racial balance (to avoid concentration of the ethnic minorities only in some schools), deviations from the average reading-age retarded proportion (to ensure that the reading-age retarded children would not concentrate in any of the schools), total road distance travelled (to try to allocate the children to the nearest secondary school), total difficulty of travel (to pick out that the same distance travelled can take a different time in different areas; measured by awarding points), deviations from the average school capacity utilization (to prevent underusage of any school), and deviations from the average sex proportions (to ensure consistent sex balance in mixed schools). The only binding constraints of the presented model ensure that all children are allocated to any secondary school and that the capacity of any secondary school would not be exceeded. All other constraints are formulated as goal constraints with penalisation of both (positive and negative) deviations. This means the model should find solution that is to the goal values as close as possible. The objective function penalises the weighted sum of all deviations. This model was applied on the Greater Reading area, which had 64 mixed primary schools and 17 comprehensive schools (11 mixed, 6 single-sex).

3.4 Allocation of offices

With allocation of offices to academic staff deal Giannikos, El-Darzi and Lees [4]. They set up an integer goal programming model to optimise relocation of offices at the University of Westminster. This university has four faculties, each composed of three or four schools, which are divided into a number of divisions. The institution is spread over 22 sites in London. The management of the university decided to consolidate activities of the university into six main campuses. This means that some amount of staff have to move to another rooms (offices). At first, academic staff was divided into three groups – lecturers, researchers, etc (group 1), heads of division (group 2) and heads of school and similar (group 3). Standard amount of required space for member of each group were defined. The model has five goals – a) assign enough offices to each school for members of group 1 and 2 (staff-count goal); b) allocate offices according to standards of required space (space standards goal); c) minimise the distances from the rooms assigned to each school and its administrative centre (distance goal); d) assign each room to only one school (exclusive usage goal); and e) minimise the number of people that have to move (relocation goal). The model also contain strict constraints – members of group 3 have to be in an office with specified space standard and in room assigned to member of group 3 can be only members of group 3. The goals were ranked according to their importance from the most important goal as follows – staff-count goal, space standards goal, exclusive usage goal, relocation goal and distance goal. This priority scheme was used in pre-emptive goal programming model (for details about pre-emptive GP please see [7]).

3.5 University Timetabling

Timetabling at universities is a problem that belongs to difficult problems. Al-Husain et al. [1] provide a sequential three-stage integer goal programming model for faculty-course-time-classroom assignments. The scheduling is divided into three stages – the faculty-course assignment stage, the courses-timeslot assignment stage, and timeslot-room assignment stage. "The inputs of every stage are translated into goals and solved according to their order of importance, where goals are given priorities according to their order of importance. The output of every stage, which represents an optimal assignment, is then fed to the next stage to act as an input." [1, p. 158]. The process continues as is shown in the Fig. 1.

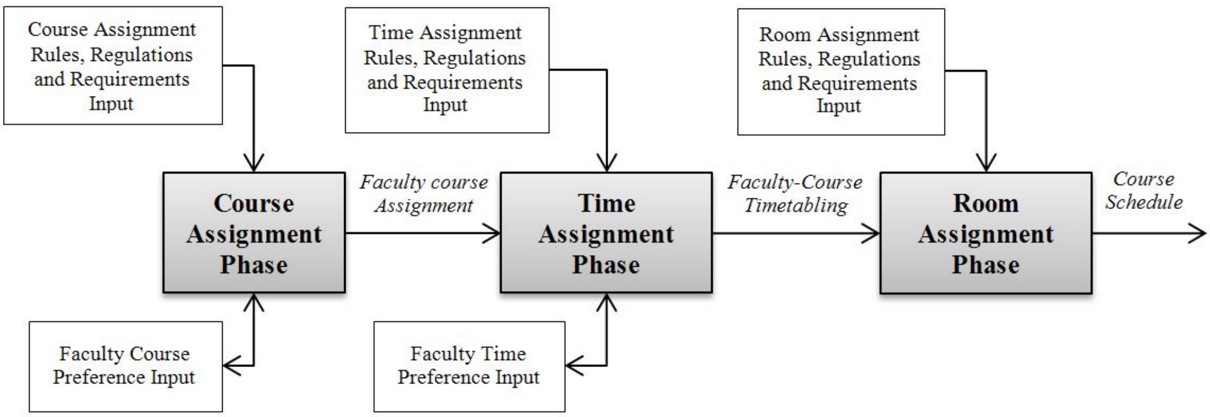

Fig. 1: Faculty Course Schedule Block Diagram and Information Flow [1]

In the stage I the courses are assigned to faculty members. The integer GP model has five goals and one strict constraint. The goals are a) limit of course loads of each faculty member; b) number of courses that should be covered by faculty members; c) each faculty member should take at least one of the College Level Courses (CLC) and d) at least one of the Major Level Courses (MLC); and e) maximisation of the total preference for each faculty member. The strict constraint is that any of the faculty members cannot take more than two sections for the same course. In the stage II the courses assigned to faculty members are assigned to time slots. This stage of model has seven goals and three strict constraints. The goals are a) number of rooms available for each time slot, the number cannot be exceeded; b) similar CLC assigned to a specific time slot in morning-time cannot exceed 2 sections for the same course; c) similar CLC assigned to a specific time slot in afternoon-time cannot exceed 1 section for the same course; d) the MLC should be 4 times more condensed during the morning-time than during the afternoon-time; e) 60% of courses should be offered during the odd days and 40% during the even days; f) 70% of courses should be offered during the morning-time and 30% during the afternoon-time; and g) faculty preferences on class times maximisation. The strict constraints are that sum of sections taught for every faculty in every specific time slot must be at most equal to 1, sum of MLC offered during a specific time slot during same day must equal at most, and sum of time slots for each section for every faculty, every course, and every section must equal 1. The stage III has only one goal and two strict constraints. In this stage the rooms are assigned to the courses. The goal is to locate each previously assigned course to a room of the right size as close as possible to the department that is offering the course. The strict constraints are that each section of a course assigned to a specific faculty and time should be located in one room only, and that each room is assigned to at most one faculty in a specific time period. In the paper [1], the model is applied on scheduling at Kuwait University, College of Business Administration.

4 CONCLUSION

Goal programming has been first formulated in the 1955. Since that time it has been improved and extended and applied to many decision problems. In this paper some important applications to various problems were presented. The aim of future research is to concentrate on the application of problem of timetabling that is prepared to be tested at author's department.

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RISK-SENSITIVE AND RISK NEUTRAL OPTIMALITY IN MARKOV DECISION CHAINS; A UNIFIED APPROACH

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Abstract
In this note we consider Markov decision chains with finite state space and compact actions spaces where the stream of rewards generated by the Markov processes is evaluated by an exponential utility function (so-called risk-sensitive model) with a given risk sensitivity coefficient. If the risk sensitivity coefficient equals zero (risk-neutral case) we arrive at a standard Markov decision chain. Necessary and sufficient optimality conditions along with equations for average optimal policies both for risk-neutral and risk-sensitive models will be presented and connections and similarity between these approaches will be discussed.

Keywords: *discrete-time Markov decision chains, exponential utility functions, risk sensitivity coefficient, connections between risk-sensitive and risk-neutral models*

JEL Classification: C44
AMS Classification: 90C40

1 INTRODUCTION AND NOTATION

In this note, we consider Markov decision processes with finite state and compact action spaces where the stream of rewards generated by the Markov processes is evaluated by an exponential utility function (so-called risk-sensitive model) with a given risk sensitivity coefficient, and slightly extend some of the results reported in [1,2, 9–12]. To this end, let us consider an exponential utility function, say $\bar{u}^{\gamma}(\cdot)$, i.e. a separable utility function with constant risk sensitivity $\gamma \in R$, where the utility assigned to the (random) outcome ξ is given by

$$\bar{u}^{\gamma}(\xi) := \begin{cases} (\text{sign}\ \gamma)\exp(\gamma\xi), \ \text{if}\ \gamma \neq 0, \\ \xi \ \ \text{for}\ \gamma = 0 \ \ (\text{the risk-neutral case}). \end{cases} \quad (1)$$

For what follows let $u^{\gamma}(\xi) := \exp(\gamma\xi)$, hence $\bar{u}^{\gamma}(\xi) = (\text{sign}\ \gamma)u^{\gamma}(\xi)$. Then for the corresponding certainty equivalent, say $Z^{\gamma}(\xi)$, since $\bar{u}^{\gamma}(Z^{\gamma}(\xi)) = \mathrm{E}[\bar{u}^{\gamma}(\xi)]$ (E is reserved for expectation), we immediately get

$$Z^{\gamma}(\xi) = \begin{cases} \gamma^{-1}\ln\{\mathrm{E}u^{\gamma}(\xi)\}, & \text{if}\ \gamma \neq 0 \\ \mathrm{E}[\xi] & \text{for}\ \gamma = 0. \end{cases} \quad (2)$$

In what follows, we consider at discrete time points Markov decision process $X = \{X_n, n = 0,1,\ldots\}$ with finite state space $I = \{1,2,\ldots,N\}$, and compact set $A_i = [0, K_i]$ of possible decisions (actions) in state $i \in I$. Supposing that in state $i \in I$ action $a \in A_i$ is chosen, then state j is reached in the next transition with a given probability $p_{ij}(a)$ and one-stage transition reward r_{ij} will be accrued to such transition. A (Markovian) policy controlling the decision process is given by a sequence of decisions at every time point. In particular, policy controlling the process, $\pi = (f^0, f^1,\ldots)$, is identified by a sequence of decision vectors $\{f^n, n = 0,1,\ldots\}$ where $f^n \in F \equiv A_1 \times \ldots \times A_N$ for every $n = 0,1,2,\ldots$, and $f_i^n \in A_i$ is the decision (or action) taken at the nth transition if the chain X is in state i. Policy π which selects at all times the same decision rule, i.e. $\pi \sim (f)$, is called stationary. We shall assume that the stream of transition rewards generated by the considered Markov decision process is evaluated by an exponential utility function (1). To this end, let $\xi_{X_0}^n(\pi) = \sum_{k=0}^{n-1} r_{X_k, X_{k+1}}$ be the (random) total reward received in the n next transitions of the considered Markov chain X if policy $\pi=(f^n)$ is followed and the chain starts in state X_0. Supposing that $X_0=i$, on taking expectation we have (E_i^π denotes the expectation if $X_0=i$ and policy $\pi=(f^n)$ is followed)

$$\bar{U}_i^\gamma(\pi,n) := \mathrm{E}_i^\pi(\bar{u}^\gamma(\xi^n)) = (\mathrm{sign}\,\gamma)\mathrm{E}_i^\pi \mathrm{e}^{\gamma\sum_{k=0}^{n-1} r_{X_k,X_{k+1}}} \quad \text{if} \quad \gamma \neq 0 \tag{3}$$

$$V_i(\pi,n) := \mathrm{E}_i^\pi(\bar{u}(\xi^n)) = \mathrm{E}_i^\pi \sum_{k=0}^{n-1} r_{X_k,X_{k+1}} \quad \text{if} \quad \gamma = 0, \quad \text{the risk neutral case.} \tag{4}$$

2 RISK-NEUTRAL CASE: OPTIMALITY EQUATIONS

To begin with, (cf. [6,8]) first observe that if the discrepancy function

$$\bar{\phi}_{ij}(w,\bar{g}) := r_{ij} - \bar{g} + w_j - w_i, \text{ for arbitrary } \bar{g}, w_i \in R, i, j \in I \tag{5}$$

then by (4)

$$V_i(\pi,n) = \bar{g} + w_i + \sum_{j\in I} p_{ij}(f_i^0)\{\bar{\phi}_{ij}(w,\bar{g}) + V_j(\pi^1, n-1) - w_j\} \tag{6}$$

$$= n\bar{g} + w_i + \mathrm{E}_i^\pi \sum_{k=0}^{n-1} \bar{\phi}_{X_k,X_{k+1}}(w,\bar{g}) - \mathrm{E}_i^\pi w_{X_n} \tag{7}$$

For what follows we introduce matrix notation. We denote by $P(f)=[p_{ij}(f_i)]$ the $N\times N$ transition probability matrix of the chain X. Recall that the limiting matrix $P^*(f) = \lim_{m\to\infty} m^{-1}\sum_{n=0}^{m-1} P^n(f)$ exists; in particular, if $P(f)$ is *unichain* (i.e. $P(f)$ contains a single class of recurrent states) the rows of $P^*(f)$, denoted $p^*(f)$, are identical.

Obviously, $r_i(f_i) := \sum_{j=1}^N p_{ij}(f_i) r_{ij}$ (resp. $\phi_i(f_i,w,\bar{g}) := \sum_{j=1}^N p_{ij}(f_i)\bar{\phi}_{ij}(w,\bar{g})$) is the expected one-stage reward (resp. expected discrepancy) obtained in state $i\in I$, and $r(f)$ (resp. $\phi(f,w,\bar{g})$) denotes the corresponding N-dimensional column vector of one-stage rewards (resp. expected discrepancies). Then $[P(f)]^n \cdot r$ (resp. $[P(f)]^n \cdot \phi(f,w,\bar{g})$) is the (column) vector of expected rewards (resp. expected discrepancies) accrued after n transitions; its ith entry denotes expectation of the reward (resp. discrepancy) obtained at time point n if the process X starts in state i. Similarly, the vector of total expected rewards

$$V(\pi,n) := \sum_{k=0}^{n-1}\prod_{j=0}^{k-1} P(f^j) r(f^k) = ng + w + \sum_{k=0}^{n-1}\prod_{j=0}^{k-1} P(f^j)\phi(f^k,w,\bar{g}) - \prod_{j=0}^{k-1} P(f^j) w \tag{8}$$

and its i-th element $V_i(\pi,n)$ is the total expected reward if the process starts in state i. Observe that for $n\to\infty$ elements of $V(\pi,n)$ can be typically infinite. Moreover, following stationary policy $\pi \sim (f)$ for n tending to infinity there exist vector of average expected rewards, denoted $g(f)$ (with elements $g_i(f)$) where $g(f) = \lim_{n\to\infty} \frac{1}{n} V(f,n) = P^*(\pi) r(f)$.

Assumption A. There exists state $i_0 \in I$ that is accessible from any state $i\in I$ for every $f\in F$, i.e. for every $f\in F$ the transition probability matrix $P(f)$ is *unichain*.

The following facts are well-known to workers in stochastic dynamic programming (see e.g. [4,7]). If Assumption A holds there exists decision vector $f^*\in F$ (resp. $\hat{f}\in F$) along with (column) vectors $w^* = w(f^*)$, $\hat{w} = w(\hat{f})$ with elements w_i^*, $\hat{w}_i$ respectively, and $g^* = g(f^*)$, resp. $\hat{g} = g(\hat{f})$ (constant vector with elements $\bar{g}(f) = p^*(f) r(f)$) being the solution of the (nonlinear) equation (I denotes the identity matrix)

$$\max_{f\in F}[r(f) - g^* + (P(f)-I)\cdot w^*] = 0, \qquad \min_{f\in F}[r(f) - \hat{g} + (P(f)-I)\cdot \hat{w}] = 0 \tag{9}$$

where $w(f)$ for $f = f^*, \hat{f}$ is unique up to an additive constant, and unique under the additional normalizing condition $P^*(f)\, w(f) = 0$. Then for

$$\phi(f, f^*) := r(f) - g(f^*) + (P(f) - I) \cdot w(f^*), \quad \phi(f, \hat{f}) := r(f) - g(\hat{f}) + (P(f) - I) \cdot w(\hat{f}) \quad (10)$$

we have $\phi(f, f^*) \le 0$, $\phi(f, \hat{f}) \ge 0$ with $\phi(f, f^*) = \phi(\hat{f}, \hat{f}) = 0$. In particular, by (9), (10) for every $i \in I$ we can write

$$\phi_i(f, f^*) = r_i(f_i) - \bar{g}^* + \sum_{j\in I} p_{ij}(f_i) w_j^* - w_i^* \le 0, \quad \phi_i(f, \hat{f}) = r_i(f_i) - \hat{\bar{g}} + \sum_{j\in I} p_{ij}(f_i) \hat{w}_j - \hat{w}_i \ge 0.$$

3 RISK-SENSITIVE MODELS: OPTIMALITY EQUATIONS

For the risk-sensitive models, let $U_i^\gamma(\pi,n) := \mathrm{E}_i^\pi(u^\gamma(\xi^n))$ and hence $Z_i^\gamma(\pi,n) = \frac{1}{\gamma}\ln U_i^\gamma(\pi,n)$ be the corresponding certainty equivalent. In analogy to (6), (7) for expectation of the utility function we get by (5) for arbitrary $\bar{g}$, $w_i \in R$, $i, j \in I$

$$U_i^\gamma(\pi,n) = \mathrm{e}^{\gamma(\bar{g}+w_i)} \sum_{j\in I} p_{ij}(f_i^0) e^{\gamma\{\bar{\phi}_{ij}(w,\bar{g}) - w_j\}} \cdot U_j^\gamma(\pi^1, n-1) \quad (11)$$

$$= \mathrm{e}^{\gamma(2\bar{g}+w_i)} \sum_{j\in I}\sum_{k\in I} p_{ij}(f_i^0) e^{\gamma\{\bar{\phi}_{ij}(w,\bar{g}) - w_j\}} \cdot \mathrm{e}^{\gamma w_j} p_{jk}(f_j^1) e^{\gamma\{\bar{\phi}_{jk}(w,\bar{g}) - w_k\}} \cdot U_k^\gamma(\pi^2, n-2)$$

$$= \mathrm{e}^{\gamma(n\bar{g}+w_i)} \mathrm{E}_i^\pi e^{\gamma\{\sum_{k=0}^{n-1} \bar{\phi}_{X_k,X_{k+1}}(w,\bar{g}) - w_{X_n}\}}. \quad (12)$$

In particular, for stationary policy $\pi \sim (f)$ assigning numbers $g(f)$, $w_i(f)$ by (5) we have

$$\bar{\phi}_{ij}(w(f), \bar{g}(f)) := r_{ij} - \bar{g}(f) + w_j(f) - w_i(f) \quad (13)$$

and (11),(12) take the form

$$U_i^\gamma(f,n) = \mathrm{e}^{\gamma(\bar{g}(f)+w_i(f))} \sum_{j\in I} p_{ij}(f_i) e^{\gamma\{\bar{\phi}_{ij}(w(f),\bar{g}(f)) - w_j(f)\}} \cdot U_j^\gamma(f, n-1)$$

$$= \mathrm{e}^{\gamma(n\bar{g}(f)+w_i(f))} \mathrm{E}_i^\pi e^{\gamma\{\sum_{k=0}^{n-1} \bar{\phi}_{X_k,X_{k+1}}(w(f),\bar{g}(f)) - w_{X_n}(f)\}}.$$

In what follows we show that under certain assumptions there exist $w_i(f)$'s, $g(f)$ such that

$$\sum_{j\in I} p_{ij}(f_i) \mathrm{e}^{\gamma r_{ij}} \cdot \mathrm{e}^{\gamma w_j(f)} = \mathrm{e}^{\gamma \bar{g}(f)} \cdot \mathrm{e}^{\gamma w_i(f)}, \text{ for } i \in I. \quad (14)$$

Now let $\rho(f) := \mathrm{e}^{\gamma g(f)}$, $z_i(f) := \mathrm{e}^{\gamma w_i(f)}$, $q_{ij}(f_i) := p_{ij}(f_i)\mathrm{e}^{\gamma r_{ij}}$ and introduce the following matrix notation $U^\gamma(\pi,n) = [U_i^\gamma(\pi,n)]$, $z(f) = [z_i(f)]$ ... N-column vectors, $Q(f) = [q_{ij}(f_i)]$ ... $N \times N$ nonnegative matrix.

Then by (14) for stationary policy $\pi \sim (f)$ we immediately have $\rho(f) z(f) = Q(f) z(f)$. Since $Q(f)$ is a nonnegative matrix by the well-known Perron-Frobenius theorem $\rho(f)$ equals the spectral radius of $Q(f)$ and $z(f)$ can be selected nonnegative. Moreover, if $P(f)$ is irreducible then $Q(f)$ is irreducible, and $z(f)$ can be selected strictly positive (cf. [3]). Finally observe that and if $P(f)$ is unichain then $z(f)$ can be selected strictly positive if the risk sensitivity coefficient γ is sufficiently close to zero.

In (14) attention was focused only on a fixed stationary policy $\pi \sim (f)$. The above facts can be extended to all admissible policies under the following

Assumption B. There exists state $i_0 \in I$ that for every $f \in F$ is accessible from any state $i \in I$, i.e. for every $f \in F$ the transition probability matrix $P(f)$ is unichain. Furthermore, if for some $f \in F$ the matrices $P(f)$ and also $Q(f)$ are reducible then state i_0 belongs to the basic class of $Q(f)$ that is unique.

If Assumption B holds we can show existence of numbers w_i^* $(i \in I), g^*$, and some $f^* \in F$ such that for all $i \in I$

$$\sum_{j\in I} p_{ij}(f_i)e^{\gamma\{r_{ij}+w_j^*\}} \le \sum_{j\in I} p_{ij}(f_i^*)e^{\gamma\{r_{ij}+w_j^*\}} = \mathrm{e}^{\gamma[g^*+w_i^*]} \tag{15}$$

or equivalently
$$\sum_{j\in I} q_{ij}(f_i)z_j(f^*) \le \sum_{j\in I} q_{ij}(f_i^*)z_j(f^*) = \rho(f^*)z_i(f^*). \tag{16}$$

Moreover, if Assumption B is fulfilled there also exist $\hat{w}_i\,(i \in I), \hat{g}$, and some $\hat{f} \in F$ such that for all $i \in I$

$$\sum_{j\in I} p_{ij}(f_i)e^{\gamma\{r_{ij}+\hat{w}_j\}} \ge \sum_{j\in I} p_{ij}(\hat{f}_i)e^{\gamma\{r_{ij}+\hat{w}_j\}} = \mathrm{e}^{\gamma[\hat{g}+\hat{w}_i]} \tag{17}$$

or equivalently
$$\sum_{j\in I} q_{ij}(f_i)z_j(\hat{f}) \ge \sum_{j\in I} q_{ij}(\hat{f}_i)z_j(\hat{f}) = \rho(\hat{f})z_i(\hat{f}). \tag{18}$$

Observe that by (17), (18) it holds for any $f \in F$

$$Q(f)z(f^*) \le Q(f^*)z(f^*) = \rho(f^*)z(f^*), \qquad Q(f)z(\hat{f}) \ge Q(\hat{f})z(\hat{f}) = \rho(\hat{f})z(\hat{f}). \tag{19}$$

Theorem. If Assumption B holds there exists decision vector $f^* \in F$ (resp. $\hat{f} \in F$) along with column vector $z(f^*)$ (resp. $z(\hat{f})$) and a positive number $\rho(f^*)$, along with $g(f^*) = \ln \rho(f^*)$, (resp. $\rho(\hat{f})$, along with $g(\hat{f}) = \ln \rho(\hat{f})$) such that for any $f \in F$ $\rho(\hat{f}) \le \rho(f) \le \rho(f^*)$ and also $g(\hat{f}) \le g(f) \le g(f^*)$.

The proof (by policy iterations) based on ideas in [5] can be found in [12].

4 NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS.

4.1. Risk-neutral case

To begin with, from Eq.(8) considered for decision vector f^* maximizing the average reward with $g = g^*$, $w = w^*$ we immediately have for policy $\pi = (f^n)$

$$V(\pi,n) := \sum_{k=0}^{n-1}\prod_{j=0}^{k-1} P(f^j)r(f^k) = ng^* + w^* + \sum_{k=0}^{n-1}\prod_{j=0}^{k-1} P(f^j)\phi(f^k, f^*) - \prod_{j=0}^{k-1} P(f^j)w^*. \tag{20}$$

Hence for stationary policy $\pi^* \sim f^*$ maximizing average reward we immediately get

$$V(\pi^*,n) = ng^* + w^* - \prod_{j=0}^{k-1} P(f^j)w^* \tag{21}$$

and (nonstationary) policy $\pi = (f^n)$ maximizes long run average reward if and only if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\prod_{j=0}^{k-1} P(f^j)\phi(f^k, f^*) = 0. \tag{22}$$

4.2. Risk-sensitive case

From Eq.(12) considered for decision vector f^* fulfilling conditions (15), (16) we immediately have for policy $\pi = (f^n)$

$$U_i^\gamma(\pi,n) = \mathrm{e}^{\gamma(g^*+w_i^*)}\sum_{j\in I} p_{ij}(f_i^0)e^{\gamma\{\bar{\phi}_{ij}(w^*,g^*)-w_j^*\}}\cdot U_j^\gamma(\pi^1,n-1) \tag{23}$$

$$= \mathrm{e}^{\gamma(ng^*+w_i^*)}\mathrm{E}_i^\pi e^{\gamma\{\sum_{k=0}^{n-1}\bar{\phi}_{X_k,X_{k+1}}(w^*,\bar{g}^*)-w^*_{X_n}\}}. \tag{24}$$

Hence for stationary policy $\pi^* \sim (f^*)$ with f^* fulfilling conditions (15), (16) we immediately get

$$U_i^\gamma(f^*,n) = \mathrm{e}^{\gamma(ng^*+w_i^*)} \mathrm{E}_i^\pi e^{\{-\gamma w^*_{X_n}\}}. \tag{25}$$

Since the state space I is finite, there exists number such that $|w_i^*| \le K$ for each $i \in I$. Hence by (2),(24),(25) we immediately conclude that

$$U_i^\gamma(\pi,n) \le \mathrm{e}^{\gamma(ng^*+w_i^*)} \cdot \mathrm{e}^{|\gamma| K}, \quad Z_i^\gamma(\pi,n) = \frac{1}{\gamma} \ln U_i^\gamma(\pi,n) \tag{26}$$

In virtue of (17),(18),(19) from (26) we can conclude that for stationary policy $\pi \sim (f^*)$ or $\pi \sim (\hat{f})$ and arbitrary policy $\pi = (f^n)$

$$\lim_{n\to\infty} \frac{1}{n} Z_i^\gamma(\pi,n) = g^* \text{ if and only if } \lim_{n\to\infty} \frac{1}{n} \ln\{\mathrm{E}_i^\pi \mathrm{e}^{\gamma \sum_{k=0}^{n-1} \bar{\phi}_{X_k,X_{k+1}}(w^*,g^*)}\} = 0 \tag{27}$$

$$\lim_{n\to\infty} \frac{1}{n} Z_i^\gamma(\pi,n) = \hat{g} \text{ if and only if } \lim_{n\to\infty} \frac{1}{n} \ln\{\mathrm{E}_i^\pi \mathrm{e}^{\gamma \sum_{k=0}^{n-1} \bar{\phi}_{X_k,X_{k+1}}(\hat{w},\hat{g})}\} = 0. \tag{28}$$

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MODELLING AND FORECASTING OF WAGES: EVIDENCE FROM THE SLOVAK REPUBLIC

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Abstract
The wages are very important indicator from the microeconomic and macroeconomic point of view. The aim of this paper is to use the econometric approach and to construct some variants of wages models. For this purpose we used classical linear model, logarithmic model and model based on principals of error correcting term (Error correction model – ECM). Finally we compare the presented models and make evaluations of the results and in order to use the best model for the forecast.

Keywords: *wages, econometric model, stationarity, ECM, forecast*

JEL Classification: C44
AMS Classification: 90C15

1 INTRODUCTION

The problematic of wages was and still it is a central point of interest of many economists. Original Phillips curve presents indirect interdependence between the rate of growth of the nominal wages and the rate of growth of unemployment. Let us mention a few studies dealing with these issues e. g [2], [3], [6]. Generally, the main determinants of wages are inflation, unemployment and labor productivity. It is not possible to separate the terms like wages, employment, inflation and output since they form together the complex area of economics. The gross wages in Slovakia achieve in international comparison of EU countries extreme low values. Taxes and charges are on average per employee 45.2 % of labor costs. According to [5] Slovakia occupied in 2011 the 16th place in the EU27 taking into account this comparison.

2 DEVELOPMENT OF WAGES – ECONOMETRIC APPROACH

Our analysis is based on quarterly data reported by the Slovak Statistical Office over the period 2000 to 2011. The models were estimated only for the period 2000 – 2010 in order to use the data from 2011 for verification purposes. The econometrical program Eviews 5.1 was used for the analyses.
We used following macroeconomic indicators:
W – average nominal wage (€ per person),
GDP – gross domestic products in current prices, (billion €),
CPI – costumer price index according to the classification of individual consumption by purpose (COICOP),
DK – dummy variable for correcting the decrease in wage growth in times of crisis,
S3 a S4 –seasonal dummy variables.

2.1 Linear and logarithmic model of wages

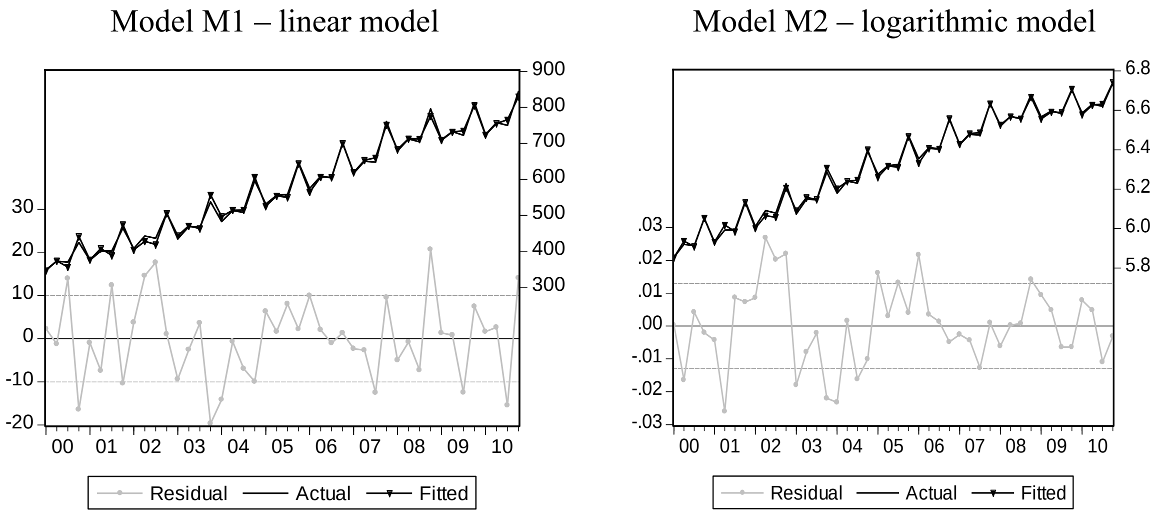
First we formulate and estimate linear model. The model was improved by the introduction of dummy variables to the model to correct the seasonal fluctuations and to slow down the increasing trend during the crisis (2009q1 – 2011q4).
Estimated linear model (M1):
$W_t = -39.84 + 27.57{*}GDP_t + 1.99{*}CPI_t + 57.72{*}S4_t - 22.11{*}S3_t + 47.47{*}DK_t$
$R2 = 0{,}996$ D-W = 2,17 BG(4) = 8,65
This model presents long run relationship between wages and output, price level, seasonal variation and crisis dummy variables. In the next step the logarithmic model was estimated.

Estimated logarithmic model (M2):

$LOG(W_t) = 2.93 + 0.65*LOG(GDP_t) + 0.36*LOG(CPI_t) + 0.09*S4_t - 0.04*S3_t +$
$+ 0.07*DK_t$

$R2 = 0{,}997$ D-W = 1,39 BG(4) = 5,39

Figures 1 Actual, fitted and residuals in M1 and M2; Source: own calculation



All estimated parameters in both models, M1 and M2 are statistically significant at the significance level of 1%. The coefficients of determination the estimated models are high. Residuals aren't correlated. The estimators fulfill the expectations concerning the signs and are economically interpreted.

2.2 ECM model of wages

Nowadays ECM is widely used also for their ability to separate short-term impact of the determinants from the long-term impact on the explained variable. The first step to the construction of the ECM is time series stationarity examination. The stationarity has been tested by augmented Dickey-Fuller test (ADF test) and Phillips-Perron test (PP test). In the ECM have been used the first differences of the variables ($D(W_t)$, $D(GDP_t)$,…) to ensure their stationarity because all the selected time series were nonstationary and integrated of first order [1]. As the integration of all-time series is the same, the cointegration concept can be applied. Johansen procedure has indicated just one cointegration vector between explanatory variable and explaining variables [7].

Next we estimated the ECM model using the original variables without the logarithmic transformation. In this variant of ECM the residual sequence of M1 (cointegration equation) was used.

Final ECM model (M3):

$D(W_t) = – 9.57 – 1.15*R\_M1_{t-1} + 38.41*D(GDP_t) – 6.92*D(CPI_t) + 95.73*S4_t –$
$– 5.15*S3_t,$

$R2 = 0{,}926$ D-W = 1,99 BG(4) = 4,62

Where R_M1_{t-1} is residual from cointegration equation model M1.

It is not possible to estimate ECM with variables in logarithmic values because Johansen procedure cointegration test indicates 3 cointegrating vectors at the 5 % level.

3 VERIFICATION OF THE MODELS AND FORECAST

All models were used to calculate forecasts for the year 2011. Then we compare forecasting values with real values of wages, see table 1.

Table 1 The forecast of wage for the year 2011; Source: own calculation

	W real	W_M1	W_M2	W_M3
2011Q1	746,00	752,00	748,96	753,03
2011Q2	781,00	792,07	792,38	790,95
2011Q3	769,00	799,92	792,30	818,44
2011Q4	848,00	868,28	889,87	912,79

Table 2 Percentage error and MAPE; Source: own calculation

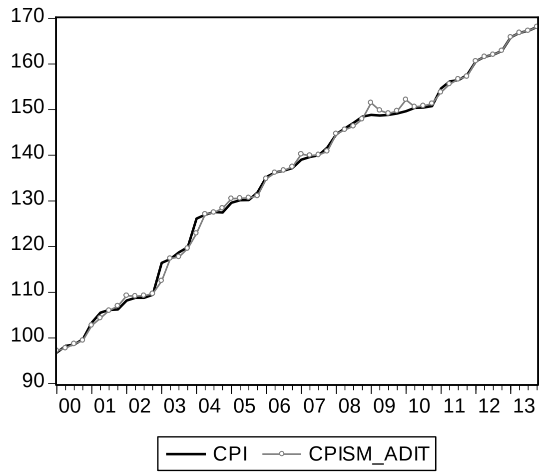
	W_M1	W_M2	W_M3
2011Q1	0,80%	0,40%	0,94%
2011Q2	1,42%	1,46%	1,27%
2011Q3	4,02%	3,03%	6,43%
2011Q4	2,39%	4,94%	7,64%
MAPE	**2,16%**	**2,46%**	**4,07%**

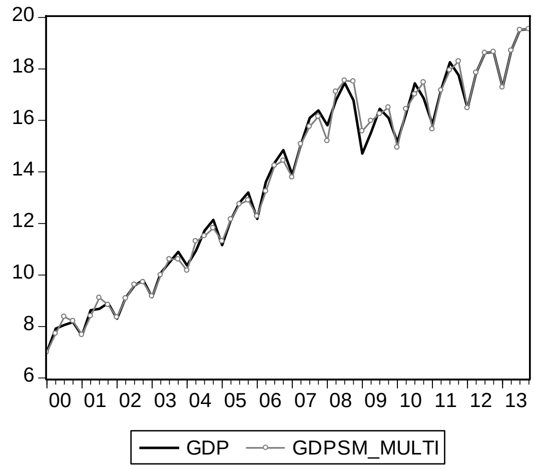
We compare the results in table 2 and the best model is model with smaller mean absolute percentage error (MAPE), it is linear model M1.

In the next step we estimate same model as M1 using the whole period: 2000q1 – 2011q4. We obtain the following model:
$W_t = -39.99 + 27.13*GDP_t + 2.03*CPI_t + 57.19*S4_t - 23.49*S3_t + 42.58*DK_t$
$R2 = 0,99$ D-W = 1,82 BG(4) = 8,752

All estimated parameters are statistically significant at the significance level of 1%. The coefficient of determination is high. Residuals aren’t correlated. Estimators fulfill the expectations concerning the signs and are economically interpreted. To obtain the value of wages for the years 2012 and 2013, we must know values of all explanatory variables. We use the methodology exponential smoothing – additive and multiplicative. More appropriate is model with smaller root mean squared error (RMSE).
We obtain following results: variable CPI – additive model (RMSE = 1,062) and GDP – multiplicative model (RMSE = 0,3121).
On the figure 2 are real values of CPI and GDP with the best exponential smoothing values.

Figures 2 Real and exponential smoothing values of CPI and GDP; Source: own calculation





Dummy variable DK for years 2012 and 2013 will be equal to 1. We assume that in this period the growth of wages will be as slow as in the time of crisis. Graphical interpretation of the real and forecasted values we can see on the figure 3 and forecasted values of wages we can see in table 3.

Figure 3 Real values and ex-ante forecasts of average nominal wages of Slovakia

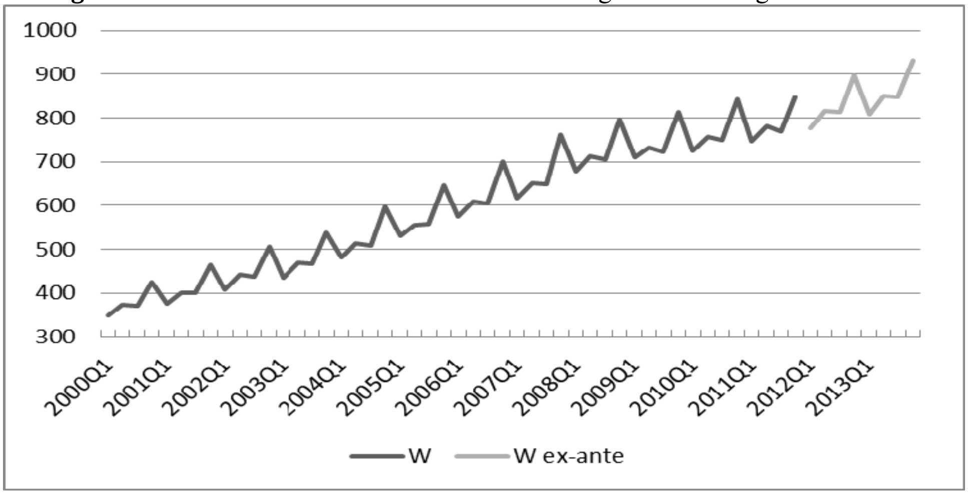


Table 3 Forecasted values of wages

	W ex-ante		**W ex-ante**
2012Q1	776,40	2013Q1	809,01
2012Q2	815,44	2013Q2	849,59
2012Q3	813,86	2013Q3	848,73
2012Q4	897,31	2013Q4	931,95

4 CONCLUSION

In our paper we presented three different econometric models of average nominal wages of Slovak republic. Two models (linear and logarithmic) were formulated regardless of stacionarity of variables and one model (ECM) took into account the stacionarity of dependent and independent variables. Models obtained were evaluated using the forecasts for year 2011. Finally on the basis of ex-post forecasts we compared models and their prognostic ability. Linear model shows to be the best model for forecast calculation for the next years.

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THE ESTIMATE OF PARAMETERS OF PRODUCTION FUNCTION OF SLOVAK ECONOMY: ECONOMETRIC ANALYSIS OF NOMINAL WAGES

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Abstract
We find our original methodology to estimate the parameters of the production function. Our models are based on the long-run first order condition of optimal behaviour of firms that states equality between wage rate and marginal product of the labour. The CES production function generates logarithmic relation between average and marginal products of the labour, while the Cobb-Douglas production function generates linear relation between average and marginal products of the labour. Both wages and average product of the labour data are available from official statistical sources. Finally we estimate specifications generated by our models. Thanks to described approach we find that the form of the production function in Slovakia has been Cobb-Douglas and average labour share has been 0.111 during 2001q1 – 2011q4 period.

Keywords: *CES and Cobb–Douglas production function, wages, average product of the labour, average labour share*

JEL Classification: C22, E23, E24, E25, O30, O52
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1 INTRODUCTION

Estimate of the values of production function has been an important topic on the quantitative research agenda for decades. The most recent approaches are [1], [6], [5] and [2].
We will provide a possibility of an estimate of the production function parameters using Slovak data of nominal average monthly wages and average product of the labour. The models are based on the long-run first order condition of optimal behaviour of firms that states equality between wage rate and marginal product of the labour. Using more general CES production function we can specify a logarithmic relation between marginal and average products of the labour and using Cobb-Douglas production function we can specify a linear relation between marginal and average products of the labour.

2 MODELS

Consider the problem of the firms in the economy, which choose capital and labour inputs (*K* and *N*) by maximizing the profit function. It is well known that the first-order condition of the problem for labour is that wage rate equals to the marginal product of the labour:

$$w_t = MPN_t \tag{1}$$

We can write the wage condition (1) in both nominal and real units. Statistical Office of the Slovak Republic publishes a corresponding data only in current prices; hence we will assume that all variables are nominal. Using this assumption, this section will be more consistent with the sections dealing with econometric analyses using Slovak data. We will show that using various known forms of the production function we can express marginal product of the labour (MPN_t) by the average product of the labour (APN_t).

Logarithmic Specification

Consider the CES Production Function in the form:

$$Y_t = A_t\left[\alpha\left(\kappa K_t\right)^\gamma + \left(1-\alpha\right)\left(\nu N_t\right)^\gamma\right]^{\frac{1}{\gamma}} \tag{2}$$

where inputs are capital and labour, K_t and N_t, $0 < \alpha < 1$ is the share parameter, γ determines the degree of substitutability of the inputs and A_t is a productivity parameter. The parameters κ and ν depend upon the units in which the output and inputs are measured and play no important role. The value of γ is less than or equal to 1 and can be $-\infty$.

If $\gamma = 1$, the production function is linear and the inputs are perfectly substitutable, if $\gamma = -\infty$, the inputs are not substitutable. Finally, if $\gamma = 0$, the production function is Cobb-Douglas. Using Cobb-Douglas production function we derive the linear specification in the next section.

We assume that the productivity parameter grows constantly with the rate g:

$$A_t = A_0 e^{gt} \tag{3}$$

Let's express the marginal product of the labour from the CES production function (2):

$$MPN_t = A_t\left[\alpha\left(\kappa K_t\right)^\gamma + \left(1-\alpha\right)\left(\nu N_t\right)^\gamma\right]^{\frac{1-\gamma}{\gamma}}\left(1-\alpha\right)\nu^\gamma N_t^{\gamma-1}$$

The log of marginal product of the labour is:

$$\log\left(MPN_t\right) = \log\left(A_t\right) + \frac{1-\gamma}{\gamma}\log\left(x_t\right) + \log\left(1-\alpha\right) + \gamma\log\left(\nu\right) + \left(\gamma-1\right)\log\left(N_t\right) \tag{4}$$

where

$$x_t = \left[\alpha\left(\kappa K_t\right)^\gamma + \left(1-\alpha\right)\left(\nu N_t\right)^\gamma\right]$$

Let's express the average product of the labour from the CES production function (2):

$$APN_t = \frac{A_t\left[\alpha\left(\kappa K_t\right)^\gamma + \left(1-\alpha\right)\left(\nu N_t\right)^\gamma\right]^{\frac{1}{\gamma}}}{N_t}$$

The log of average product of the labour is:

$$\log\left(APN_t\right) = \log\left(A_t\right) + \frac{1}{\gamma}\log\left(x_t\right) - \log\left(N_t\right) \tag{5}$$

Combining (5) and (4) we can derive the relation between logs of marginal and average marginal product of the labour in the form:

$$\log\left(MPN_t\right) = \log\left(1-\alpha\right) + \gamma\log\left(\nu\right) + \gamma\log\left(A_t\right) + \left(1-\gamma\right)\log\left(APN_t\right) \tag{6}$$

Substituting logarithm of the marginal product of the labour, (6) and logarithm of the productivity growth, (3) into the logarithm of the wage condition (1) we gain the logarithmic specification for wages in the form:

$$\log\left(w_t\right) = \beta_0 + \beta_1\log\left(APN_t\right) + \beta_2 t + u_{gt} \tag{7}$$

where the stochastic term u_{gt} captures the demand, monetary and real shocks and satisfies standard assumptions of the econometric model and coefficients are:

$$\beta_0 = \log\left(1-\alpha\right) + \gamma\log\left(\nu\right) + \gamma\log\left(A_0\right) \tag{8}$$

$$\beta_1 = 1-\gamma \tag{9}$$

$$\beta_2 = \gamma g \tag{10}$$

Unfortunately the logarithmic specification (7) is overidentified, nevertheless an estimation of the specification will provide several information about production function. From the coefficient β_1 we know about substitution abilities of the inputs, the inverse value of the coefficient is the average elasticity of the substitution of the inputs. Moreover we can determine the type of the production function from this value. Using both coefficients β_1 and β_2, we can compute the average productivity growth rate, g.

Linear specification

Consider more specific case, where $\gamma = 0$ and the production function is in the Cobb-Douglas form:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{11}$$

Let's go back to the derivation of the marginal product of the labour using average product of the labour. Let's express the marginal product of the labour from the Cobb-Douglas production function (11):

$$MPN_t = \left(1-\alpha\right) A_t K_t^{\alpha} N_t^{-\alpha} \tag{12}$$

and the average product of the labour from the same function (11):

$$APN_t = A_t K_t^{\alpha} N_t^{-\alpha} \tag{13}$$

Combining (12) and (13) we express the linear relation between marginal and average product of the labour in the form:

$$MPN_t = \left(1-\alpha\right) APN_t \tag{14}$$

Substituting the marginal product (14) of the labour into the wage condition (1) we gain the linear specification for wages in the form:

$$w_t = \delta_0 + \delta_1 APN_t + u_{nt} \tag{15}$$

where the stochastic term u_{nt} captures the demand, monetary and real shocks and satisfies standard assumptions of the linear econometric model. We assume the value of the coefficient δ_0 equals to zero and the coefficient δ_1 equals to the average labour share, $\delta_1 = (1 - \alpha)$ and the reminder to unity is the average capital share.

3 DATA AND METHODOLOGY

We estimate the coefficients of the logarithmic specification (7) and, since our estimate is, $\gamma = 0$ (as it will turn up) – the production function is Cobb-Douglas – we also estimate the coefficients of the linear specification (15).

The quarterly data are gathered from the SLOVSTAT database. We measure wages as nominal average monthly wages. The average product of the labour is the ratio of the gross domestic product in current prices and employment. We seasonally adjusted data using the TRAMO procedure. The available range of data is 1995q1 – 2011q4, however – estimating the logarithmic specification (7) – we detected a structural breakpoint in 2000q1; we use the Chow test [3] to verify the breakpoint in 2000q1 and the corresponding F-statistics is 4.558. Therefore we use data with the range 2000q1 – 2011q4. Both wages and average product of the labour and logs of them are integrated of order one. This statement is supported by the augmented Dickey-Fuller test, Phillips-Perron test and correlograms of the variables (see [7] for more details). Therefore we use the Error Correction Model to estimate the coefficients of both specifications (7) and (15) (see Engle and Granger, [4]). In all estimates autocorrelation is tested using Durbin-Watson test, Ljung-Box Q-statistics and Breusch-Godfrey LM test (see [8] for more details). Autocorrelation was eliminated using the dynamic specification.

4 RESULTS

Logarithmic Specification

The cointegration relation of the specification (7) is:

$$\log(w_t) = -0.581 + 0.139\log(APN_t) - 0.001t + 0.913\log(w_{t-1}) - 0.032UM03q1$$
$$(0.149)(0.032) \qquad (0.000)\ (0.045) \qquad (0.006)$$

The *UM03q*1 is dummy variable with one in 2003q1 and zeros in other observations. The numbers in parenthesis placed below estimated coefficients are estimates of standard errors. The R-squared coefficient of the equation is 0.999. The steady state values of the coefficients β_0 to β_2 defined in (8)-(10) can be computed using condition $\log(w_t) = \log(w_{t-1})$ and they are in the Table 1. As we calculate the coefficients using nonlinear restriction, we use the χ^2 statistic with degree of the freedom of 1 restriction and 48 observations minus 5 coefficients (i.e. 43) to verify the significance of the coefficients.

Table 1 The estimates of the coefficients of the logarithmic specification (7)

	β_0	β_1	β_2
coefficient	-6.702	1.598	-0.012
$\chi^2(1,43)$	2.453	8.022	1.168
p-value	0.117	0.005	0.280

The coefficients β_0 and β_2 are not statistically significant; the coefficient β_1 is significant on one-percent level. The insignificancy of the trend in the specification (7) and value of the coefficient β_1 close to 1 implies that $\beta_1 = 1$ and $\gamma = \beta_2 = 0$. We didn't reject the hypothesis $\beta_1 = 1$, the corresponding $\chi^2(1,43)$ statistic was 1.123. However the estimate is sort of problematic, because the trend and the average product of the labour are linearly correlated. The multicolinearity comes from the fact, that average product of the labour grows as the productivity grows. In our assumption it grows by a constant rate. Actually the correlation coefficient between the average product of the labour and the trend is 0.978.

If the $\beta_1 = 1$ and so $\gamma = 0$ than $\beta_2 = 0$ in the specification (7) and we can omit the trend from the specification. We estimated the equation in the form:

$$\log(w_t) = -0.380 + 0.175\log(APN_t) + 0.825\log(w_{t-1})$$
$$(0.116)\ (0.039) \qquad (0.035)$$

The R-squared is 0.999. The steady state values of the coefficients β_0 to β_2 defined in (8)-(10) with corresponding $\chi^2(1,45)$ statistics are in the Table 2. The value of the $\chi^2(1,45)$ statistics for the test of the hypothesis $\beta_1 = 1$ is 0.001 and we again do not reject it. The elasticity of the substitution of the inputs has been unity and so Cobb-Douglas production function has explained a transformation of inputs into output in the Slovakia during the analysed period.

Table 2 The estimates of the coefficients of the logarithmic specification (7) without trend

	β_0	β_1	β_2
coefficient	-2.173	0.999	0
$\chi^2(1,45)$	53.163	998.237	-
p-value	0.000	0.000	-

We made the series of deviations between actual values of the dependent variable and values of the dependent variable fitted by the steady state econometric model as:

$$e_{gt} = \log(MPN_t) - \hat{\beta}_0 - \hat{\beta}_1\log(APN_t)$$

where a shed denotes that a values of the corresponding coefficient is estimated (from Table 2). After adjusting the series from the structural breakdown in the 2009q1 – 2011q4 period caused by the financial crisis, using the Augmented Dickey-Fuller test we state that the series is stationary. The error correction specification is:

$$\Delta\left[\log\left(w_t\right)\right] = \underset{(0.098)}{0.252}\Delta\left[\log\left(APN_t\right)\right] + \underset{(0.111)}{0.620}\left[\log\left(w_{t-1}\right)\right] - \underset{(0.010)}{0.030}UM03q1 - \underset{(0.095)}{0.215}e_{gt-1}$$

Linear Specification

Since the Slovak production function has been Cobb-Douglas, we may estimate the specification (15). The cointegration relation of the specification is:

$$w_t = \underset{(3.438)}{1.683} + \underset{(0.003)}{0.017}APN_t + \underset{(0.031)}{0.844}w_{t-1}$$

The steady state values of the coefficients δ_0 and δ_1 with corresponding $\chi^2(1,45)$ statistics are in the Table 3. The R-squared coefficient of the equation is 0.997, the estimate of the labour share in period 2000q1 – 2011q4 has been 0.111. We cannot reject the hypotheses that the constant δ_0 equals to zero which coincides with our theory.

Table 3 The estimates of the coefficients of the linear specification (15)

	δ_0	δ_1
coefficient	10.791	0.111
$\chi^2(1,45)$	0.216	1157.586
p-value	0.642	0.000

We made the series of deviations between actual values of the depended variable and values of the depended variable fitted by the steady state econometric model (e_{nt}). Using the Augmented Dickey-Fuller test we state that the series is stationary and it was used to estimate the error correction specification:

$$\Delta w_t = \underset{(0.006)}{0.015}\Delta apn_{t-1} - \underset{(2.828)}{15.434}um03q4 + \underset{(2.822)}{9.400}um07q4 - \underset{(0.141)}{0.314}e_{nt-1} + \underset{(0.174)}{0.377}u_{nt-1} + \underset{(0.172)}{0.576}u_{nt-2}$$

We used the dummy variables eliminating observations in 2003q4 and 2007q4 and the first and the second order autocorrelation coefficients.

5 CONCLUSION

The elasticity of the substitution of the inputs has been unity and so Cobb-Douglas production function has explained a transformation of inputs into output in the Slovakia during 2000q1 – 2011q4 period. The average labour share in Slovakia has been 0.111.

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THE PRESENCE OF HYSTERESIS ON THE LABOUR MARKETS IN SELECTED COUNTRIES

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Abstract
The presented article deals with the problem of hysteresis in unemployment. Its goal is to compare the labor markets of selected countries and to answer the question whether the development in these countries can be regarded as hysteresis. The introductory part of the paper acquaints the reader with the basic concept of hysteresis and its application to the labor market in the form of the Phillips curve hysteresis. In the practical part the presence of hysteresis in unemployment in selected European countries is empirically tested. The presence of hysteresis and the ability to describe the hysteresis phenomenon is studied by applying the unit root test, single equation model and vector-autoregressive. The conclusion provides a summary of test results and outlines the basic features of economic policy implications in unemployment hysteresis.

Keywords: *Hysteresis in unemployment, unit root test, single equation model, VAR model*

JEL Classification: C44
AMS Classification: 90C15

1 INTRODUCTION

Hysteresis can be understood as a phenomenon in which a system progresses from one equilibrium level to another as a response to a shock. Individual levels are dependent on previous values therefore hysteresis is often confused with inertia.
Microeconomic nature of hysteresis phenomenon is clarified by so-called hysteresis mechanisms that, in case of the labor market, are in the form of insider-outsider hypothesis, of the role of long-term unemployed and of the effect of recession on the capital stock and investment. Hypothesis insider-outsider is trying to describe the behavior and the role of trade unions in wage negotiations. Group of long-term unemployed people have weaker response to stimuli associated with the recovery of the economy (new jobs) and also reacts similarly to government support which tend to hit group of the short-term unemployed. The effect of recession on the capital stock and investment influences the labor market in the form of layoffs, i.e. increasing the number of unemployed. During the recession investment spending is reduced and therefore new jobs are not created.

1.1 Hysteresis Phillips curve

Theory is based on the original Phillips curve that describes the relationship of growth rate of nominal wages and unemployment. Later, the growth rate of nominal wage was replaced by inflation and the concept of the natural rate of unemployment and Non-Accelerating Inflation Rate of Unemployment (NAIRU) has been widely discussed.

2 EMPIRICAL ANALYSIS

Analysis was performed on data from 37 countries and five economic clusters. The data consist of unemployment rate in the form of monthly data obtained from Eurostat and OECD, inflation was expressed using monthly data with annual variation in the Harmonized Index of Consumer Prices (HICP) published by Eurostat and OECD. The data were analyzed for all countries in three different time periods. The first period is from January 1998 to December 2011 and consists of 168 observations. The second is a shortened version of the first and corresponds to the period from January 2008 to December 2011 (48 observations). This corresponds approximately to the period of economic crisis, when the change in the labor market can be expected and therefore it can be interesting to see the presence of hysteresis in unemployment. The last study period consists of quarterly data from first quarter 1998 to fourth quarter 2011, which represent 43 observations.

2.1 Procedures for detecting hysteresis in labor markets

The first test procedure is a standard unit root test. This test has been subjected to the time series of unemployment and basically responds to the question whether the time series is stationary or not. In terms of hysteresis mechanism this test belongs to insider-outsider hypothesis, according to which the time series of unemployment should be matched to nonstationary stochastic process, most likely to random walk. More specifically, the augmented Dickey and Fuller test (ADF test) was used and length of delay is automatically adjusted according to the Schwarz information criteria. In this case a cross in the table 1 implies that on the 5% significance level, we can conclude the presence of a unit root. In terms of hysteresis in unemployment, we can conclude that the simple version of the insider-outsider hypothesis is relevant to explain the nature of unemployment in the country.

The second test assumes using of single equation model and its verification. Single equation model of hysteresis corresponds to the expression of the Phillips curve. Verification of a particular model lies in its ability to adequately describe the reality, acquire significant parameters and also in elimination of autocorrelation and other ailments. After verification of the model the presence of hysteresis is monitored easily. The coefficients adherent to u_t and u_{t-1} have opposite signs and their proportion is approximately one. We assume full hysteresis if the ratio is one. A specific example of the Czech Republic and other countries is given below. A cross in the table 1 therefore indicates that the model describes well the fact that the individual coefficients are significant and, of course, their ratio is about one.

The third test consisted of vector-autoregressive model (VAR) based on the shape of the Phillips curve hysteresis. Using VAR models emerged from two impulses. The first is the actual shape of the hysteresis Phillips curve which resembles a general VAR model for two variables and lag. Another frequent complaint is the presence of autocorrelation in single equation model and the need for its elimination. Verification of the model consisted of the application of Johansen procedure. The issue of VAR models is beyond the scope of this article, further information can be found in Hušek [4] or Greene [3]. A cross in the table 1 means that the presence of hysteresis in model can be with a probability of 95% confirmed. Small frequency of positive results may be caused by need of satisfying several conditions for the model verification.

3 THE PRESENCE OF HYSTERESIS IN EUROPEAN COUNTRIES

The table 1 clearly shows the analysis results of monitoring the presence of hysteresis in unemployment in selected countries. A cross (x) in the table means that the test results conclude the presence of hysteresis in unemployment. The last column of the table simply shows the number of positive results of hysteresis in unemployment. The number of four or five concludes to hysteresis in unemployment in the country, however, smaller number indicates possible hysteresis in selected country and requires further studies.

Table 1 Results of analysis of the presence of hysteresis in unemployment

Country/Period	1998m1 - 2011m12			2008m1 - 2011m12			1998q1 - 2011q4			Sum of positive results
	Unit root test	Single equation model	VAR model	Unit root test	Single equation model	VAR model	Unit root test	Single equation model	VAR model	
Austria	x	x	-	x	-	-	-	-	-	3
Belgium	-	-	-	-	-	-	x	-	-	1
Bulgaria	-	-	-	x	-	-	-	-	-	1
Brazil	-	-	x	-	-	-	x	-	-	2
Canada	x	-	-	-	-	-	x	-	-	2
Chile	x	-	-	-	x	-	x	-	-	3
Cyprus	-	-	-	x	-	-	-	-	-	1
Czech Republic	-	x	-	-	x	-	x	x	x	5
Germany	-	-	x	-	-	-	-	-	-	1
Denmark	-	-	-	-	-	-	-	-	-	0
Euro area (17 countries)	x	x	-	-	-	-	-	-	-	2
Spain	-	x	-	-	x	-	-	x	-	3
Estonia	-	x	-	-	x	-	x	x	-	4
European Union (27 countries)	x	x	-	-	x	-	-	-	-	3
Finland	-	-	-	-	-	-	-	-	-	0
France	x	-	-	-	-	-	x	-	-	2
G7	x	-	-	-	-	-	-	-	-	1
Greece	-	x	-	-	x	x	-	-	-	3
Croatia	-	-	-	-	-	x	x	-	-	2
Hungary	-	-	-	-	-	x	-	-	-	1
Ireland	-	x	x	-	x	-	-	x	x	5
Italy	-	-	-	-	-	-	-	-	-	0
Japan	-	-	-	-	-	-	-	-	-	0
Korea	x	-	x	x	x	-	x	-	-	5
Lithuania	x	-	x	-	-	-	x	x	-	4
Luxembourg	x	-	x	-	-	-	-	-	x	3
Latvia	-	x	-	-	x	-	-	x	-	3
Malta	-	-	-	-	-	-	-	-	-	0
Mexico	x	-	x	-	-	-	x	-	x	4
Netherlands	x	-	-	-	-	-	x	-	-	2
Norway	-	-	-	-	-	-	-	-	-	0
OECD - Europe	x	x	-	-	-	x	-	x	-	4
OECD - Total	x	-	-	-	-	-	-	-	-	1
Poland	-	-	-	x	-	-	x	-	-	2
Portugal	-	-	x	-	x	-	-	-	x	3
Russian Federation	x	-	-	-	-	-	x	-	-	2
Romania	x	-	-	-	-	-	-	-	-	1
Sweden	x	-	-	-	-	-	x	-	-	2
Slovak Republic	-	-	-	-	x	-	-	-	-	1
Slovenia	-	-	-	x	-	x	-	-	-	2
United Kingdom	-	-	-	-	-	x	-	-	-	1
United States	x	-	-	-	-	-	-	-	-	1

3.1 Unit root test

Insider-outsider hypothesis of hysteresis mechanism is connected with a unit root test. As we mentioned above, in the case of presence of unit root in time series of unemployment, the mechanism can be explained by the influence of trade unions on unemployment. In Slovenia in the period from 1998 to 2011, the time series data do not contain a unit root, although the shorter period from 2008 to 2011 does contain the unit root. This can be interpreted in a way that the year 2008 increased the power of trade unions in wage bargaining. During this period, countries were affected by the economic crisis, which could be a trigger for employee dismissing and consequently it could weld insiders in their position against outsiders and as a result one of the hysteresis mechanisms would be fulfilled.

3.2 Single equation model

The following model is based on the shape of the hysteresis Phillips curve of the relationship between p_t (inflation), u_t (unemployment) where α, β are parameters of the model and u_t^* is NAIRU:

$$p_t = \alpha p_{t-1} + \beta(u_t - u_t^*) \tag{1}$$

If α is equal to 1 then NAIRU is in steady state when p_t=p_{t-1}. Hysteresis is found if u_t^* depends on lagged rate of uneployment u_{t-1} and other microeconomic determinants (z_t):

$$u_t^* = \eta u_{t-1} + z_t \tag{2}$$

Then the joint model has formula:

$$p_t = \alpha p_{t-1} + \beta(u_t - \eta u_{t-1} - z_t) \tag{3}$$

For our analysis we use this form of the model:

$$p_t = \alpha p_{t-1} + \beta_1 u_t + \beta_2 u_{t-1} + \varepsilon \tag{4}$$

The derivation of the model can be found in Gordon [2]. The final model explains the evolution of unemployment in the Czech Republic in the period from January 1998 to December 2011 giving the following results:

$$\underset{}{\hat{p}_t} = \underset{(0{,}01)}{0{,}95p_{t-1}} - \underset{(0{,}18)}{0{,}65u_t} + \underset{(0{,}18)}{0{,}66u_{t-1}} \tag{5}$$

The figures in parentheses represent standard deviations of parameter estimates and the coefficient of determination R^2 is 0.96. The model thus captures reality reasonably and therefore we can evaluate on this basis the presence of hysteresis. In the case of full hysteresis the proportion of estimated parameters corresponding to the variables of unemployment rate (u_t, u_{t-1}) should be equal to one as well as these parameters should have, according to [2], opposite signs. As it is shown in (5), these requirements were fulfilled, so we can establish the presence of hysteresis in unemployment in the Czech Republic from 1998 to 2011. The presence of hysteresis in unemployment in the single equation model for the Czech Republic is significant also in other time periods.

3.3 Vector-autoregressive model

Vector-autoregressive (VAR) model is simultaneous equations model based on hysteresis Phillips curve. We mention the theory background for this model in the section 2.1. The estimation results for Ireland in the period from 1998 to 2011 were as follows:

$$\hat{u}_t = \underset{(0{,}06)}{1{,}59u_{t-1}} - \underset{(0{,}06)}{0{,}59u_{t-2}} - \underset{(0{,}03)}{0{,}07p_{t-1}} + \underset{(0{,}03)}{0{,}07p_{t-2}} + \underset{(0{,}06)}{0{,}002} \tag{6}$$

$$\hat{p}_t = \underset{(0{,}12)}{-0{,}34u_{t-1}} + \underset{(0{,}12)}{0{,}33u_{t-2}} + \underset{(0{,}07)}{1{,}46p_{t-1}} - \underset{(0{,}07)}{0{,}5p_{t-2}} + \underset{(0{,}11)}{0{,}15} \tag{7}$$

Similarly as in the single equation model, we are interested in the ratio of unemployment rates. In this case the first two coefficients from equation (7) correspond to this ratio, both have opposite sign and the ratio is approximately one. For equation (6) R^2 reaches 0.99 and for equation (7) R^2 is equal to 0.98. We therefore conclude similar results as in single equation model for the Czech Republic. In our analysis we verify the presence of hysteresis in Ireland based on the monthly data and the VAR model.

4 CONCLUSION

We conclude in our analysis the presence of hysteresis in the Czech Republic, Ireland and Korea. The hysteresis phenomenon is also presented on the labor markets of Estonia, Lithuania, Mexico and in the cluster of European OECD members. At the other end are Nordic countries Denmark, Finland and Norway as well as Italy, Malta and Japan where the labor markets do not show any sign of hysteresis based on analysis.

The theory of hysteresis or hysteresis phenomenon in the labor markets is still in terms of economic theory only marginal and many economic policy makers are not aware of it. The results of our analysis show that the impact on labor markets is not negligible. For this reason it needs wider popularization and further development of the theory, particularly in the countries concerned.

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A DYNAMIC ANALYSIS OF CAUSALITY BETWEEN PRICES ON THE METALS MARKET

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Abstract
The aim of the paper is to analyse causality between the prices of four different metals: gold, silver, platinum and copper. The analysis conducted in this paper is a dynamic one, and the data used consist of monthly prices of futures contracts traded from the period January 2000 – January 2012. The assessment of causality was carried out with the use of rolling regression applied to VAR models, which allowed for the assessment of the stability of relations between metal prices. The results obtained indicate that causality changed in the period analysed. Initially the price of copper was the Granger cause of the prices of the remaining metals, while in the later period the price of platinum became the Granger cause of the prices of the remaining metals. Past prices of gold and silver did not improve the forecasts of prices of other metals.

Keywords: *Granger causality, rolling regression, Toda -Yamamoto tests, metals market*

JEL Classification: C32, G15, O13, Q37
AMS Classification: 91B84, 62M10

1 INTRODUCTION

Gold, platinum and silver are the most popular precious metals. For centuries they have been primarily used to make jewellery, whereas nowadays they also play an important part in various industries, for example, gold is used in electronics, telecommunication and aviation, silver is used in the electronic and electrical industries (mobile phones, computer hardware), and, to a smaller extent, in photography, while platinum is mostly used in the chemical and petrochemical industries as a catalyst and in the motor industry for building catalytic converters, as well as in the electronic and electrical industries. Copper, obtained together with silver, is used in the electronic and electrical industries (wires), as well as the building, machine, and telecommunication industries. In recent years these metals have become an important means of the saurisation, and are now frequently used as an investment through, for instance, the purchase of gold bars or the purchase of Exchange traded funds. Such funds allow for the investment in metals without the need of their actual possessing. (For example, in January 2005 The iShares Gold Trust and in April 2006 The iShares Silver Trust were created).

World gold, silver and copper prices are listed on several stock exchanges, the most important of them being the Commodity Exchange in New York (COMEX) and London Metal Exchange (LME). The largest exchanges of actual trade and trade in futures of platinum are the New York Mercantile Exchange (NYMEX) and London Platinum&Palladium Market.

The vital role of metals on financial markets and in various industries has found its reflection in science. Ciner [2] examined long-term trends between the prices of gold and silver futures contracts traded in the 1990s and concluded that there was no relationship between gold and silver prices. Lucey and Tully [8] confirmed that in the 1990s there was no relationship between gold and silver prices, but found such a relationship in the period 1987-2002. The differences between those two periods may result from the changing nature of the demand patterns for gold versus silver. Sari et al. [10] noticed a strong relationship between gold and silver, while the highly cyclical copper appeared to be nearly independent of the movements in the prices of oil, gold and silver. Hammoudeh and Yuan [6] analysed the relations between oil prices as a determinant, two precious metals (gold and silver) and one base metal (copper) in the USA based on the daily data from the period January 2, 1990 – May 1, 2006. They concluded that gold and silver had similar volatility persistence globally, but there was no leverage effect in gold and silver prices. Soytas et al. [12], analysing the period between May 2, 2003 and March 1, 2007 found out that the world oil prices had no predictive power of other precious metal prices, the interest rate nor the exchange rate market in Turkey. Sari et al. [11] examined the co-movements and information transmission among the spot prices of four precious metals (gold, silver, platinum, and palladium), the oil price, and the US dollar/euro exchange rate. They found evidence of a weak long-term equilibrium relationship but strong feedback in the short run. Papież and Śmiech [9] conducted research connected with relations between the prices of the most important primary fuels on the European market from October 2001 till May 2011, and their analysis indicated a long-term equilibrium between those prices.

The aim of the paper is to analyse causality between the prices of metals in the period 2000-2011. The analysis is dynamic and based on monthly data. The evaluation of causality will be carried out with the use of rolling regression which allows for the assessment of the stability of the relations between metal prices. The following hypotheses will be verified: causal relations between metal prices are not stable and undergo constant changes; gold, nowadays mostly used as security on the financial markets, is a financial instrument which has no influence on the prices of other metals; the situation on the platinum and copper markets, i.e. metals commonly used in industry, influences the prices of other metals.

2 METHODOLOGY

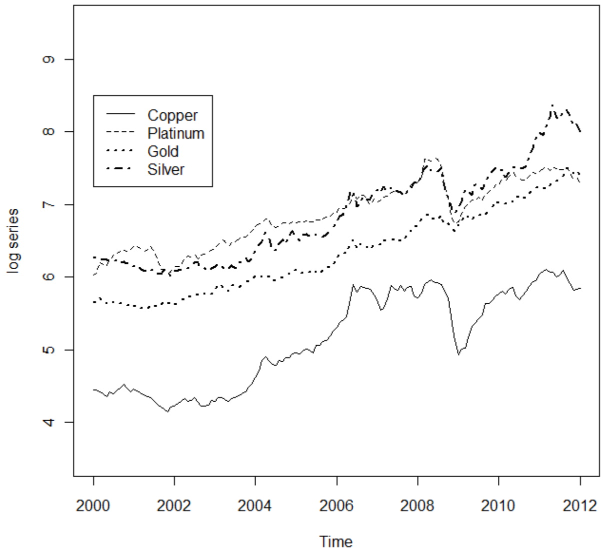
In order to test causality in unrestricted VAR models we conducted Toda and Yamamoto [13] and Dolado and Lutkepohl [4] procedures, which involved a modified Wald test and did not require pretesting for cointegration properties of the system. The idea of the preposition is to artificially augment the true lag length (say, p determined by information criterion) of a VAR model by maximal order of integration of the processes (d). The next step requires the estimation of VAR with (p+d) order, ignoring the coefficients of the last d lagged vector, and testing the restrictions on the first k coefficient matrices by a standard Wald test.

Granger [5] pointed out that the most important problem facing researches today is the structural instability. A common technique to assess the constants of model parameters and the stability of variable relations is to compute parameter estimates over a rolling window containing a fixed sample size [1]. The number of observations must by fixed to compromise between two conflicts: the degree of freedom of estimation results requires a large sample size, while the potential structural change of model requires a small sample size. The analysis and recommendations can be found in Zapata and Rambaldi [14]. We applied the fixed window rolling regression to the level VAR model. This means that we ran a series of regressions with a fixed sample size (a fixed window size). Every time the number of lags in VAR models was determined by AIC. Then, the parameters of VAR(p+d) models were estimated, and, finally, the modified Wald statistics was used to test the Granger causality.

3 EMPIRICAL RESULTS

The data used in this study consisted of monthly prices of gold, platinum, silver and copper futures contracts traded on the COMEX and the NYMEX. The analysis of causalities were conducted using the monthly time series data from the period January 1, 2000 – January 1, 2012, which yielded 145 observations. The data were taken from the stooq.pl basis. Taking into consideration the increase of variance over time, price series were logarithmed, and the results are presented in Figure 1.

Figure 1 Logarithms of prices of copper, platinum, gold and silver in the period January 2000 – January 2012.



The prices of all metals increased in the period analysed, which indicated that drifts or deterministic trends should be taken into consideration within unit root tests. Using the augmented Dickey–Fuller (ADF) method [3] and KPSS test for stationarity [7], it was verified that all metal prices were first-difference stationary I(1) (Table 1). The number of lags in the test was established using AIC criterion.

Table 1 ADF and KPSS tests of unit roots in monthly data (January 2000 – January 2012)

Variables	ADF tests in level data		ADF tests in first-differenced data		KPSS tests in level data		KPSS tests in first-differenced data	
	C	C +T	C	C +T	C	C +T	C	C +T
LCopper	-1.08 (1)	-2.488 (1)	-7.564 (0)	-7.537 (0)	6.263 (1)	0.584 (1)	0.103 (1)	0.103 (1)
LGold	1.64 (2)	-2.837 (0)	-9.684 (1)	-10.07 (1)	7.134 (1)	0.972 (1)	0.283 (1)	0.030 (1)
LPlatinum	-1.61 (1)	-3.384 (1)	-7.864 (0)	-7.852 (0)	6.466 (1)	0.341 (1)	0.068 (1)	0.038 (1)
LSilver	0.49 (0)	-3.124 (4)	-8.988 (1)	-9.126(1)	6.622 (1)	0.566(1)	0.189 (1)	0.046 (1)

Notes: C- constant; C+T - Constant and trend. The 95% critical values are −2.86 for ADF with constant and -3.41 for ADF with a constant and trend. The asymptotic critical values for 5% is 0.463 and for 1% is 0.739 for KPSS with constant and for 5% is 0.146 and for 1% is 0.216 for KPSS with a constant and trend.

Conducting the analysis within the rolling regression requires obtaining the window size. In the analysis we used monthly observations from a 5-year period, which means that each window consisted of 60 observations (we followed the recommendations of [14]). The number of lags in VAR models is presented in Figure 2. The horizontal axis indicates a starting point of window (of analysis). The first value represents the number of lags for the model estimated for the period from January 2000 to January 2005. The last one represents the number of lags in VAR estimated for the window January 2007 – January 2012. The most frequent fixed number of lags in the model was 3 or 4.

Figure 2 Number of lags in VAR models for the consecutive windows of the analysis

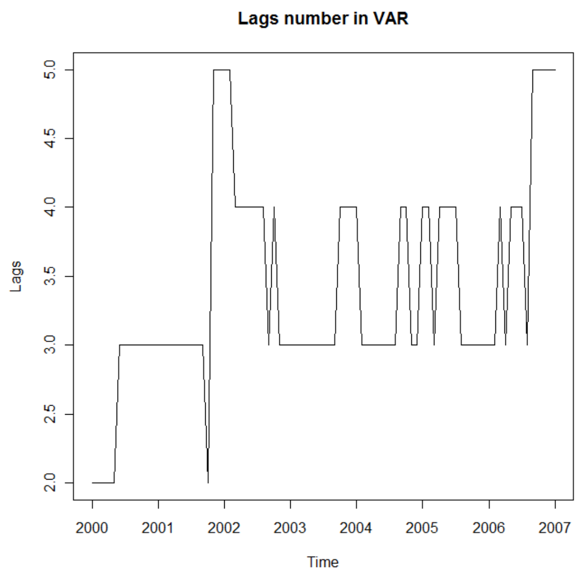
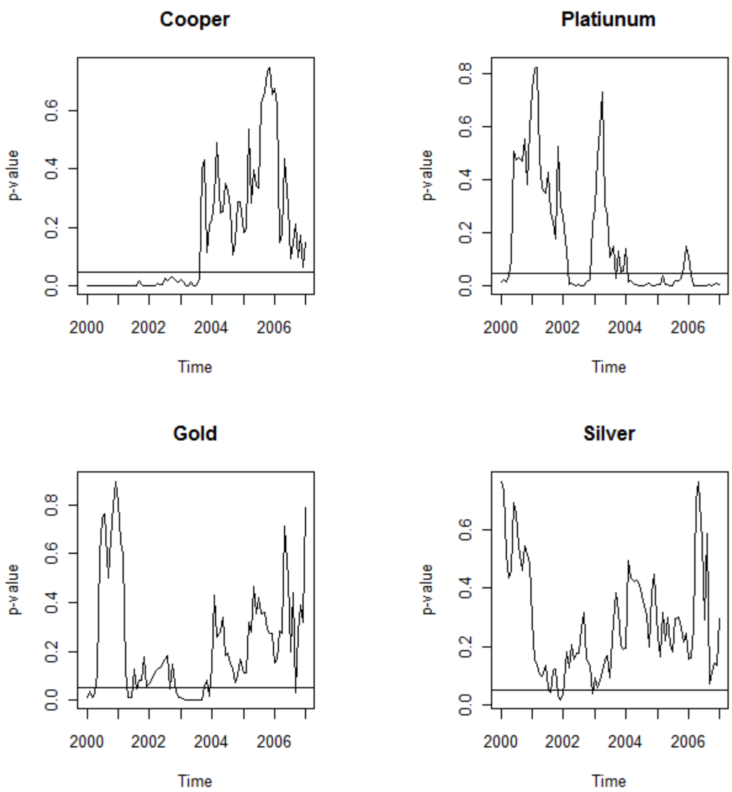


Figure 3 presents p-value for Granger causality tests. The horizontal line in the chart indicates the significance level of 5%. The values of p-value below this line mean that in a given window the price of the analysed metal is the Granger cause for the vector of the remaining metal prices. The values of p-value above this line mean that with such a significance level there are no justifications for rejecting the hypothesis of the lack of causality between prices of a given metal

and the vector of the remaining prices. The results presented in Figure 3 indicate that causal relations between the elements of the system were not stable in time.
The results of the analysis indicate that the price of copper was the Granger cause of the prices of other metals from the beginning of the period analysed till the end of 2008. In the later period past information regarding the price of copper did not improve the forecasts of others metals prices. A different situation could be observed in case of the platinum prices. In the initial period (with the exception of several first windows) p-value significantly exceeded the 5% significance level, which means that in this period the platinum price was not the Granger cause of other metals prices. In the models obtained from 2004 onward the price of platinum was the Granger cause of prices of the remaining metals. Thus, the forecast of the prices of copper, gold and silver were improved by the values of the platinum price. In the majority of periods analysed, past gold prices did not improve forecasts of the remaining metals with the exception of the windows beginning in 2003 (the period analysed 2003 - 2008). The results of the Granger causality test showed that in the period 2000-2011 the price of silver was not the Granger cause of the prices of other metals.

Figure 3 The value of p-value of Wald test. Null hypothesis states the lack of Granger causality.



4. CONCLUSION

The analysis conduced indicated significant structural changes on the international metals markets. The role of particular variables in the analysed system changed with time. The results obtained justified using a dynamic approach, such as rolling regression.

The results of causality tests revealed that metals playing an important role in the industry (i.e. platinum and copper) are more useful in forecasting than other metals. During the initial period, copper carried a forecasting force, and later this function was taken over by platinum. Silver and gold were not the Granger causes of other metals within the model used. It is worth noticing that the results obtained are characteristic for monthly metal prices, and the more frequent data may display different characteristics.

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DICTATORSHIP VERSUS MANIPULABILITY DILEMMA

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Abstract
By voting we mean the following pattern of collective choice: There is a set of alternatives and a group of individuals. Individual preferences over the alternatives are exogenously specified and are supposed to be orderings. The group is required to choose an alternative on the basis of stating and aggregating of individual preferences, or to produce a ranking of alternatives from the most preferred to the least preferred. In this paper concepts of manipulation as strategic voting (misrepresentation of true preferences) and dictatorship (voting procedure leads in all cases to social rankings that are identical with rankings of an individual) are investigated. The connection between Arrow's and Gibbard-Satterthwaite's theorems is discussed from the viewpoint of dilemma between dictatorship and manipulability: there exists no voting procedure which satisfies at the same time non-dictatorship and non-manipulability.

Keywords: *Arrow's theorem, dictatorship, Gibbard-Satterthwaite theorem, manipulability, strategic voting*

JEL Classification: D71
AMS Classification: 91F10

1 INTRODUCTION

Social choice analyses have benefited from the use of mathematics. Mathematic modeling has made its way from economics into the other social sciences, often accompanied by the same controversy that raged in economics in the 1950's. The reasons for this expansion are several. First, mathematics makes communication between researchers succinct and precise. Second, it helps make assumptions and models clear; this bypasses arguments in the field that are a result of different implicit assumptions. Third, proofs are rigorous, so mathematics helps avoid mistakes in the literature. Fourth, its use often provides more insights into the models. And finally, the models can be applied to different contexts without repeating the analysis, simply by renaming the symbols (Brams 2008, Schofield 2004, Turnovec 2010).

Considerable social choice literature exists regarding manipulability of voting procedures (Taylor 2005, Taylor and Pacelli 2008). Manipulability is usually understood as misrepresenting voters' preferences to get more beneficial outcome of voting (Gibbard 1973, Satterthwaite 1975, Gärdenfors 1979): On the basis of an information (or a hypothesis) about rankings of other voters and corresponding social rankings (defined by used voting rule) the voter submits such ranking, that maximizes her "utility" from resulting social ranking.

Two famous social choice theorems are related to the problems of dictatorship and manipulability. While the Arrow's "impossibility" theorem (Arrow 1963)is usually associated with non-existence of non dictatorial social preference function, the Gibbard-Satterthwaite's theorem shows that any non-dictatorial non-degenerate social choice function is manipulable. In fact, many authors observe that the both theorems are closely related (Reny, 2000). In this paper we try to reformulate Arrow's and Gibbard-Satterthwaite's theorems from the viewpoint of dilemma between dictatorship and manipulability.

2 MODELS OF VOTING AND MANIPULATION

By voting we mean the following pattern of collective choice: There is a set of alternatives and a group of individuals. Individual preferences over the alternatives are exogenously specified and are supposed to be orderings. The group is required to choose an alternative on the basis of stating and aggregating of individual preferences, or to produce a ranking of alternatives from the most preferred to the least preferred.

Let U denotes a finite set, then by $\Pi(U)$ we denote the set of strict linear orders, or (strict) rankings, on U, by $\Pi^*(U)$ we denote the set of weak linear orders, or (weak) rankings, on U.

Let N denotes the set of n individuals (voters), U a universe of alternatives (finite set of cardinality m), and $Z \subseteq U$ is a subset of U of cardinality $t \leq m$. By $\Pi^n(Z)$ we denote n-fold Cartesian product of $\Pi(Z)$, and by $\Pi^{*n}(Z)$ n-fold Cartesian product of $\Pi^*(Z)$. An element

$$\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_n) \in \Pi^n(Z)$$

is called a preference profile on Z. A preference profile on Z is a set of individual preference relations π_i on Z with one and only one preference relation for each individual $i \in N$.

By voting problem we mean the following: given N, U, $\boldsymbol{\pi} \in \Pi^n(Z)$ and some set A of social choice rationality axioms, select $Z \in 2^U$ and for selected Z find: a) either social ordering $\pi_0 \in \Pi^*(Z)$ satisfying A, b) or $z_0 \in Z$ satisfying A.

If Z is fixed, a function f: $\Pi^N(Z) \to Z$ will be called a *social choice function*, while a function F: $\Pi^N(Z) \to \Pi^*(Z)$ will be called a *social preference function*.

Let $z \in Z$ and $\boldsymbol{\pi} \in \Pi^n(Z)$, then by $z(\pi_i)$ we denote order number of alternative z in i'th individual ordering π_i (1 for top alternative, 2 for second alternative etc.), and by $\beta(z, \pi_i) = t - z(\pi_i)$ so called Borda score of z in the i'th voter's ranking.

We say that a social choice function $f(Z, \boldsymbol{\pi})$ has a property of: *Pareto efficiency* if whenever alternative x is at the top of every individual i's ranking, π_i, then $f(Z, \boldsymbol{\pi}) = x$; *monotonicity* if whenever $f(Z, \boldsymbol{\pi}) = x$ and for every individual i and every alternative y the ranking π'_i ranks x above y if π_i does, then $f(Z, \boldsymbol{\pi}') = x$; *dictatorship* if there is an individual i such that $f(Z, \boldsymbol{\pi}) = x$ if and only if x is at the top of i's ranking π_i; *non-degeneracy* if for every $x \in Z$ there exist a preference profile $\boldsymbol{\pi} \in \Pi^n(Z)$ such that $f(Z, \boldsymbol{\pi}) = x$.

We say that a social preference function $F(\boldsymbol{\pi}, Z)$ has a property of: *Pareto efficiency* if whenever alternative a is ranked above b according to each π_i, then a is ranked above b according to $F(\boldsymbol{\pi}, Z)$; *independency of irrelevant alternatives* if whenever the ranking of a versus b is unchanged for each i = 1, 2, …, n when individual i's ranking changes from πi to $\pi' i$; then the ranking of a versus b is the same according to both $F(\boldsymbol{\pi}, Z)$ and $F(\boldsymbol{\pi}', Z)$; *dictatorship* if there is an individual i such that $F(\boldsymbol{\pi}, Z) = \pi_i$ (one alternative is ranked above another in the social ranking whenever the one is ranked above the other according to the individual ranking π_i); *strategic voting manipulability* if there exists a preference profile $\boldsymbol{\pi}$, a subset of individuals $K \subset N$ and a preference profile $\boldsymbol{\pi}'$ such that $\pi'_i = \pi_i$ for $i \in N \backslash K$, $F(\boldsymbol{\pi}, Z) = \pi_0$, $F(\boldsymbol{\pi}', Z) = \pi'_0$, and for all $i \in K$ it holds that $d(\pi_i, \pi_0) < d(\pi_i, \pi'_0)$.

To illustrate concepts of strategic voting and strategic nomination we shall use the Borda social choice function and Borda social preference function. Let $N(x, y, \boldsymbol{\pi})$ be number of voters who prefer x to y ($x, y \in Z$), given a preference profile $\boldsymbol{\pi}$. Function

$$\phi(x,\boldsymbol{\pi}) = \sum_{y \in Z} N(x,y,\boldsymbol{\pi})$$

measures how many times a candidate x was preferred to the other candidates y for all $y \in Z$.

We shall use Borda's social choice (social preference) function

$$f(Z,\boldsymbol{\pi}) = \{x : x = \arg\max_{z \in Z} \ \phi(z,\boldsymbol{\pi})\}$$

selecting the candidate that received the maximum total number of votes in all pair-wise comparisons to other candidates. Borda's social preference function ranks the alternatives in order of the values of the function $\phi(x, \pi)$.

Consider three alternatives {A, B, C} and 90 voters divided into four groups with identical preferences of each group: (1) of 20 voters, (2) of 20 voters, (3) of 20 voters, and (4) of 30 voters. In Table 1a we provide preference profile $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_3)$:

Table 1a

(1) π_1	(2) π_2	(3) π_3	(4) π_4
20	20	20	30
A	A	C	B
B	C	B	A
C	B	A	C

In Table 1b see the matrix of pair-wise comparisons for preference profile π.

Table 1b

	A	B	C	
A	0	40	70	110
B	50	0	50	100
C	20	40	0	60

Assuming sincere voting the Borda winner is A, Borda social ranking [A, B, C].
If the group (4) of 30 voters with honest orderings π_4 decides to misrepresent their true preferences by π'_4 and the other voters are following their true preferences, we move to the preference profile $\pi = (\pi_1, \pi_2, \pi_3, \pi'_4)$, see Table 2a:

Table 2a

(1) π_1	(2) π_2	(3) π_3	(4) π'_4
20	20	20	30
A	A	C	B
B	C	B	C
C	B	A	A

The matrix of pair-wise comparisons (Table 2b):

Table 2b

	A	B	C	
A	0	40	40	80
B	50	0	50	100
C	40	40	0	60

The Borda winner is B, the Borda social ranking [B, (A,C)]. There exists an incentive for strategic voting of the group (3).

3 DICTATORSHIP VERSUS MANIPULABILITY

Two famous social choice theorems are related to the problems of dictatorship and manipulability. While the Arrow's "impossibility" theorem is usually associated with non-existence of non dictatorial social preference function, the Gibbard-Satterthwaite theorem shows that any non-dictatorial non-degenerate social choice function is manipulable. In fact, many authors observe that the both theorems are closely related (Reny, 2000). In this part of the paper we try to reformulate Arrow's and Gibbard-Satterthwaite theorems in terms of manipulability.

Gibbard-Satterthwaite's theorem 1: If card (Z) ≥ 3, and social choice function f(Z, π) satisfies Pareto efficiency, non-dictatorship and non-degeneracy, then f(Z, π) is manipulable.

Gibbard-Satterthwaite's theorem 2: If card (Z) ≥ 3, and social choice function f(Z, π) satisfies Pareto efficiency, monotonicity and non-degeneracy, then f(Z, π) is dictatorial.

Arrow's theorem 1: If card (Z) ≥ 3, and the social preference function F(Z, π) satisfies Pareto efficiency and non-dictatorship, then F(Z, π) is manipulable.

Arrow's theorem 2: If card (Z) ≥ 3, and social preference function F(Z, π) satisfies Pareto efficiency and independence of irrelevant alternatives, then F(Z, π) is dictatorial.
Monotonicity is a special case of independency of irrelevant alternatives. A social choice function is non manipulable if and only if it is monotonic. A social preference function is not manipulable if and only if it satisfies the independence of irrelevant alternatives. The Gibbard-Satterthwaite's theorem is a special case of Arrow's theorem.

4 CONCLUDING REMARKS

Since the Arrow's result was first published in 1951, a vast literature has grown on impossibility theorem. The great debate started about practical political conclusions from the Arrow's result. In the same way, the Gibbard-Satherthwaite theorem raised questions about how people will behave in making social decisions. For example: what sorts of strategies will they adopt when they are all voting dishonestly? What is the equilibrium when everybody is "cheating"? Theorems imply the problem of political legitimacy: in a world in which voters are misrepresenting their preferences, it is difficult to say that the outcome selected is "right", "correct" or "legitimate". Suppose for instance that candidate A wins an election process in which there were several other candidates, and the people "slightly misrepresented" their "true" preferences. Is the candidate A in such case a legitimate people's choice?
The question is: why so strictly insist on "non-manipulability"? Voting is a game, with, perhaps, imperfect information. The outcome depends on choices made by many independent decision makers. Strategic rationality of voters is a standard assumption in theory of decision. Any manipulable social choice function is better than dictatorship.
While the great achievement of Arrow and Gibbard-Satterthwaite impossibility theorem was to state the problem and to show that this sort of problems can be analyzed in a general framework of the application of rigorous mathematical methods to the social sciences, there is no reason for resigning on analyzing of particular social choice procedures and considering all of them equally bad or unusable.

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GAMES IN TWO-STAGE SUPPLY CHAINS: A CRITICAL REVIEW OF EXISTING MODELS AND THEIR POSSIBLE EXTENSIONS

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Abstract
The process of price and/or quantity selection in a supply chain consisting of independent agents provides a straightforward application of game-theoretical models, a fact that had long been neglected in the literature. Even though the first economic analyses of economic structures in supply chains emerged already in the 1950s (most notably Spengler, 1950), and many other studies followed in the next three decades, most of the economic literature before the 1990s fails to fit the analyses in a game-theoretical framework, which renders the comparison of the individual models rather difficult. A game-theoretical comparison of some of the existing models has recently been carried out independently by Lau and Lau (2005) and Kogan and Tapiero (2007). Our paper tries to compare the models reviewed in both of these sources, together with some alternatives that either have been neglected in both of these studies or have appeared in literature since. We conclude with proposing possible extensions to the existing models.

Keywords: *supply chains, successive monopoly, game theory, bargaining.*

JEL Classification: L14
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1 INTRODUCTION

With the advent of globalization, supply chains (SC) started to play an ever-increasing role in industrial organization. It is therefore only natural that supply chains analysis witnessed an upsurge in the economics literature since the mid-20th century. One of the first rigorous analyses of supply chains is attributed to Spengler (1950), who used a formal (albeit simplistic) model that studied pricing decisions inside a three-stage supply chain; the aim was to show that vertical integration in a SC can actually lead to higher consumer surplus, and hence should not be viewed as illegal by the regulators. It was particularly the interest in merger regulation that motivated most of the early papers on SCs; most of these papers were extensions and generalizations to Spengler's original setting (we outline some of these extensions in §5).

As a SC consists of several independent (but interconnected) agents, game theory (GT) seems to be the natural modeling framework for SC analysis. However, in the times when the first SC models were created, GT was mostly dealing with zero-sum games, which have hardly any connection to the SC problems. Later, when considerable work was done in various types of non-zero-sum conflicts in GT, the SC literature had already developed many sophisticated SC models, and most researchers failed to identify and/or explain the GT foundations of their models. This situation gradually changed with increasing general apprehension of GT, and several researchers have recently pointed out that most SC models make use of a very limited range of gaming processes that can be plausibly used to describe SC negotiations. It could prove very important to revise the results of the existing studies in the context of other game structures than those originally (and sometimes perhaps unintentionally) used by their authors.

The aim of this paper, however, is much less ambitious. We focus only on a very basic SC setting, namely a simple two-stage SC operating in a linear and deterministic environment, and we restrict ourselves to studying possible forms of *single-period pricing games*. Similar analyses have been recently carried out by Lau and Lau (2005) and Kogan and Tapiero (2007). Our paper tries to (1) compare the models reviewed in both of these sources, together with some alternatives that either have been neglected in both of these studies or have appeared in literature since, (2) provide a more complete and accurate taxonomy of possible pricing models, (3) discuss the plausibility of each of the individual gaming processes, and, finally, (4) outline the main directions in which the modelling framework could be extended.

2 MODELLING FRAMEWORK: A SIMPLE TWO-STAGE SC

A simple two-stage SC consists of two agents, the upstream and downstream firm, denoted *U* and *D* respectively. Typically, *U* is the manufacturer and *D* is the dealer, but many other appropriate interpretations can be given as well. *U* produces *q* units of output with a unit cost of *c*, and sells them at wholesale price *w* to *D*, who then retails the output to the end consumers at price $p = w + m$, where *m* is *D*'s mark-up to the wholesale price. The relation-ship between *p* and *q* is given by the end consumers' (linear and deterministic) demand function $q(p) = a - bp$, where *a* and *b* are positive constants. Thus, the profits of the players can be expressed as

$$\pi_U = (w - c)q \;\; = (w - c)(a - bp), \tag{1}$$

$$\pi_D = mq \qquad = m(a - bp). \tag{2}$$

The aim of both players is to achieve maximum profit through well-chosen pricing strategies. The main characteristics observed for a concrete solution are the resulting consumer surplus (which is proportional to output quantity), the SC's total profit, $\pi_U + \pi_D$, and the profits ratio, π_U / π_D. As a benchmark, the former two are compared to the *centralized* or *integrated* supply chain, which arises when both agents are owned by a single entity, who therefore maximizes the total profit, $\pi_U + \pi_D = (p - c)(a - bp)$. It can easily be shown (using first-order conditions) that the maximum total profit in the integrated chain and the related output quantity are

$$\pi_I = \tfrac{1}{4b}(a - bc)^2, \qquad q_I = \tfrac{1}{2}(a - bc). \tag{3}$$

3 CLASSIFICATION OF POSSIBLE PRICING GAMES

In the following list, we review the different pricing games from the two sources mentioned in the introduction, along with some alternatives that can be found in the recent literature.

- *(Kogan and Tapiero, 2007)*. The obvious authors' aim is to focus primarily on multi-period and differential games, and single-period games are taken only as the necessary first step in model development. Nevertheless, in their review of alternative single- and multi-period pricing games, the authors distinguish two basic model types: the simultaneous pricing game and the sequential game in the form of a leader-follower (or Stackelberg) model, with *U* being the leader (this is the form equivalent to Spengler's original model).
- *(Lau and Lau, 2005)*. Without explicitly stating it, Lau and Lau focus solely on different Stackelberg sequential games. The first one, denoted [mS] in the paper, is the same Stackelberg model as the one from the previous paragraph. Then, the authors describe a reversed leader-follower model, with the leader being *D* instead of *U*. However, there is a caveat in this model: while the upstream firm naturally chooses the wholesale price if it moves first, the first-moving downstream firm can commit itself to either the end price *p*, or the profit margin *m*, which can furthermore be specified either as an absolute (or fixed-dollar) mark-up, as it was presented in §2, or as a fixed-percentage markup to the wholesale price (so that *D* picks $g > 0$ on her move, and then charges $m = gw$ after *w* has been announced by *U*). Lau and Lau argue that the fixed-percentage markup is both a common real-life practice and an assumption of other (non-GT) economic models. As the game with *D* moving first and picking *p* seems not very plausible (since *D* cannot produce any profit, unless by fault of *U*), Lau and Lau neglect it and assume only the mark-up models, denoted [rS$] and [rS%] in their paper. Besides these three models, Lau and Lau indentify another four gaming processes – neither of which we consider appropriate. (In the following explanation of our objections, we refer to Lau and Lau's notation, which we cannot describe here due to space limitation, we refer the reader to Lau and Lau's paper.) Firstly, the [f$] and [f%] models are not to be considered real games, as *D* does not pick any strategy (her mark-up is given and fixed throughout the game) – so these situations really only model *U*'s decision-making situation. Secondly, the gaming processes of [fO$] and [fO%] are not properly defined; if they were, it

would be obvious that the [fO$] and [fO%] models merely describe the process of backward induction in the [rS$] and [rS%] games, and therefore represent the same games – a fact being considered by Lau and Lau as a separate result of their calculations.

- *Non-cooperative bargaining games*. Recently, Kvasnička et al. (2011) applied Rubinstein's well-known bargaining model to the process of price and quantity selection in a two-stage SC, arguing that this bargaining scheme will often be the closest approximation to the real-life pricing negotiations. Their model can easily be transformed into a pricing-only game.
- *Cooperative bargaining games*. In last decade, there several attempts have been made to apply the methods of cooperative GT to SC analysis. A good overview, albeit for slightly different class of SC models than our simple two-stage SC pricing model, is given in (Nagarajan and Sosic, 2008); out of the many methods presented here, it is especially the Nash bargaining model that can readily be adapted to our setting, because most other models draw upon the coalition theory, and make therefore sense with more than two players.

All in all, our taxonomy classifies six types of pricing games; our notation uses a similar square-bracket convention as that in (Lau and Lau, 2005): *simultaneous pricing* [S], *upstream-first pricing* [U], *downstream-first fixed-dollar mark-up* [D$], *downstream-first fixed-percentage mark-up* [D%], *Rubinstein bargaining pricing model* [RB] and *Nash Bargaining pricing model* [NB]. As will be shown in section 3, these different games give quite different results for our stylized model. A natural question arises: under which circumstances are the individual games plausible? Or, in other words, what justifies using a particular game for modeling purposes? This question has no definite answer, yet we present some preliminary insights.

Throughout the SC literature, [U] has undoubtedly been the most widely used one (see also section 4), typically without any justification. The intuition behind this model may lie in the chronological sequence of the real-life process: first, the intermediate production is manufactured and priced, and only then enters the dealer and proceeds with the rest. However, Kvasnička et al. (2011) argue that [U] implicitly relies on an unrealistic assumption that D behaves as a price-taker, and that their model [RB] resolves this inconsistency. Their model, on the other hand, relies on the fact that the deal that was struck in bargaining can eventually be enforced (e.g. through a binding contract), i.e. the players will not regard their negotiations as cheap talk; if they will, and the real-life process has the chronological nature outlined above, [U] seems to be the plausible gaming scheme. Bresnahan and Reiss (1985) try to identify other conditions under which [U] is credible; moreover, they show that the automobile industry in the U.S. in late 1970s fulfilled most of these conditions, and verify their analyses using empiric data.

As noted by Lau and Lau (2005), [D$] and [D%] are the schemes that resemble the real-life managerial practices, for which they should not be overlooked; however, the cheap-talk argument given above applies here as well. If prices at both stages are set up either simultaneously or secretly, and all pre-play negotiation can be for some reason regarded as cheap talk, the most plausible model is [S]. On the contrary, [NB] should be used if pre-play negotiation can be enforced by a binding contract.

4 RESULTS SUMMARY

Table 1 below shows the resulting solutions (equilibrium, subgame-perfect equilibrium, or bargaining) of the six games – [S], [U], [D$], [D%], [RB] and [NB] – in the two-stage SC from §2. We focus here on the characteristics outlined in §2, namely the ratio of individual profits π_U / π_D (which measures the relative power of the agents) and two measures of the SC's allocation efficiency: the ratio of the chains output quantity (q) to q_I, and the ratio $(\pi_U + \pi_D) / \pi_I$, where q_I and π_I were defined in (3). In all games but [D%], these ratios are invariant with respect to model parameters a, b, and c. For [D%], parameter-dependent closed-form solutions exist, but they are not very informative due to their complexity; Lau and Lau (2005) demonstrate the results using numeric examples. In case of the [RB] and [NB] model, we consider (for simplicity) the symmetric versions with equal discount rates (δ, $0 < \delta < 1$) and bargaining powers of both players. In the [RB] model, the profit ratio π_U / π_D depends on who starts the bargaining process – there's a slight first-mover advantage. Overall, the results indicate the following:

- If bargaining can be assumed (no matter whether the cooperative or non-cooperative version), the SC has the same allocation efficiency as its integrated counterpart. The allocation efficiency decreases more rapidly in the leader-follower games than in the simultaneous pricing model.
- All sequential games exhibit a first-mover advantage. In [U], [D$] and [RB], this advantage is symmetric: both *D* and *U* gain the same advantage when they switch to the leader position. In [U] and [D$], however, this only seems to be the special case due to the linear functional form, as indicated by [D%]; in [RB], this result is general, as shown by Kvasnička et al (2011).

Table 1: Summary of the results of different pricing games

Game	π_U / π_D	q / q_I	$(\pi_U + \pi_D) / \pi_I$
[S]	1	2/3	8/9
[U]	2	1/2	3/4
[D$]	1/2	1/2	3/4
[D%]	typically < 1/2	< 1	< 1
[RB]	$1/\delta$ (*U* starts) or δ (*D* starts)	1	1
[NB]	1	1	1

5 POSSIBLE EXTENSIONS

The modeling framework presented here can be extended in many ways:

- *Different SC shapes.* The two-stage model can be naturally extended into a multi-stage SC (sometimes called *successive monopoly*), as e.g. in (Machlup and Taber, 1960). The stages do not have to follow in a successive manner; instead, they might form an assembly tree, cf. (Carr and Karmarkar, 2005) or (Zouhar, 2007). Greenhut and Ohta (1979) were among the first to replace the (monopolistic) agents with groups of competing oligopolists (thus forming a *successive oligopoly* SC); Corbett and Karmarkar (2001) give an in-depth analysis of multi-stage successive oligopolies. Note that all papers mentioned in this paragraph so far only employ the [U] game (or a straightforward extension thereof). The only other model that has been employed (to our knowledge) in the multi-stage setting is the [NB] model, used by Nagarajan and Sosic (2008) for the assembly SCs. It could be worth exploring the results of the remaining games for these SCs.
- *Different demand functions*. The linear demand function can be replaced with either a general (typically continuous and downward-sloping) one – as in (Bresnahan and Reiss, 1985), a specific one – such as the iso-elastic curves in (Lau and Lau, 2005) or (Zouhar, 2009), or a variety of specific curves (Tyagi, 1999). For some games, preliminary analyses of two-stage SCs give results that differ dramatically from the linear version – these results should be checked in the context of multi-stage models as well.
- *Other extensions*. Other extensions include stochastic demand models (Nagarajan and Sosic, 2008, and Kogan and Tapiero, 2007), models of incomplete/asymmetric information (Esmaeili and Zeephongsekul, 2010, and Lau and Lau, 2007), multi-product models (Bresnahan and Reiss, 1985), and multi-period models.

6 CONCLUSIONS

In this paper, we presented a review of existing pricing games in supply chains, provided their systematic classification and suggested some ideas for further research. The latter mainly consist in applying some the games in an extended SC setting (i.e., a more complex one than the simple two-stage model analyzed in this paper). One important point was neglected here, though, and that is the need to verify the theoretical results with empiric data. To our knowledge, the only study that attempted this is (Bresnahan and Reiss, 1985). Therefore, we consider it very important that thorough empirical analyses be carried out in future in order to support the theoretical results and contribute to the discussion about the plausibility of different gaming processes.

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